Other Subordination Results for Fractional Integral Associated with Dziok-Srivastava Operator

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Abstract: In this paper we have discussed differential subordination properties associated with the fractional integral by using Dziok-Srivastava operator.

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1. INTRODUCTION

Denote by $U$ the unit disc of the complex plane, $U = \{z \in \mathbb{C} : |z| < 1\}$ and $H(U)$ the space of holomorphic functions in $U$.

Let $A_n = \{f \in H(U) : f(z) = z + a_{n+1} z^{n+1} + \ldots, z \in U\}$ with $A_1 = A$ and $H[a, n] = \{f \in H(U) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots, z \in U\}$ for $a \in \mathbb{C}$ and $n \in \mathbb{N}$.

If $f$ and $g$ are analytic functions in $U$, we say that $f$ is subordinate to $g$, written $f \prec g$, if there is a function $w$ analytic in $U$, with $w(0) = 0$, $|w(z)| < 1$, for all $z \in U$, such that $f(z) = g(w(z))$, $z \in U$. If $g$ is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subseteq g(U)$.

Let $\psi : \mathbb{C} \times U \to \mathbb{C}$ and $h$ an univalent function in $U$. If $p$ is analytic in $U$ and satisfies the (second-order) differential subordination

$$\psi(p(z), z p'(z), z^2 p''(z), z) \prec h(z), \quad z \in U,$$

then $p$ is called a solution of the differential subordination. The univalent function $q$ is called a dominant of the solutions of the differential subordination, or more simply a dominant, if $p \prec q$ for all $p$ satisfying (1.1).

A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants $q$ of (1.1) is said to be the best dominant of (1.1). The best dominant is unique up to a rotation of $U$.

Definition 1.1 ([3]) For $f \in A$, the Dziok-Srivastava operator is defined by

$$H^I_m(\alpha_1, \alpha_2, \ldots, \alpha_l; \beta_1, \beta_2, \ldots, \beta_m) : A \to A,$$

$$H^I_m(\alpha_1, \alpha_2, \ldots, \alpha_l; \beta_1, \beta_2, \ldots, \beta_m) f(z) = z + \sum_{j=2}^{m} \frac{\Gamma(j+1)}{\Gamma(j+1+\lambda)} (\beta_1)_{j-1} \ldots (\beta_m)_{j-1} (j-1)! a_j z^{j+\lambda}, \quad \alpha_i \in \mathbb{C}, \quad i = 1, 2, \ldots, l, \quad \beta_i \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}, \quad k = 1, 2, \ldots, m,$$

where $(x)_j$ is the Pochhammer symbol defined, in terms of the Gamma function by

$$(x)_j = \frac{\Gamma(x+j)}{\Gamma(x)}.$$
Differentiating we obtain

\[ z(D_z^+H_m^t[\alpha_i]f(z))' = \alpha_i D_z^+H_m^t[\alpha_i+1]f(z) - [\alpha_i-(1+\lambda)]D_z^+H_m^t[\alpha_i]f(z). \]  

(1.6)

**Lemma 1.1** (Miller and Mocanu [2]) Let \( g \) be a convex function in \( U \) and let \( h(z) = g(z) + n\alpha zg(z) \), for \( z \in U \), where \( \alpha > 0 \) and \( n \) is a positive integer.

If \( p(z) = g(0) + p_nz^n + p_mz^m + \ldots \), \( z \in U \), is holomorphic in \( U \) and

\[ p(z) + \alpha zp'(z) < h(z), \quad z \in U, \]

then

\[ p(z) < g(z), \quad z \in U, \]

and this result is sharp.

**2. MAIN RESULTS**

**Theorem 2.1** Let \( g \) be a convex function, \( g(0) = 0 \) and let \( h \) be the function \( h(z) = g(z) + \lambda zg(z) \), for \( z \in U \).

If \( f \in \Lambda \) and satisfies the differential subordination

\[ (D_z^+H_m^t[\alpha_i]f(z))' < h(z), \quad \text{for } z \in U, \]

(2.1)

then

\[ \frac{D_z^+H_m^t[\alpha_i]f(z)}{z} < g(z), \quad \text{for } z \in U, \]

and this result is sharp.

**Proof.** Consider \( p(z) = \frac{D_z^+H_m^t[\alpha_i]f(z)}{z} \), for \( z \in U \).

Let \( D_z^+H_m^t[\alpha_i]f(z) = zp(z) \), for \( z \in U \).

Differentiating we obtain \( (D_z^+H_m^t[\alpha_i]f(z))' = p(z) + zp'(z) \), for \( z \in U \).

Then (2.1) becomes

\[ p(z) + zp'(z) < h(z) = g(z) + \lambda zg(z), \quad \text{for } z \in U. \]

By using Lemma 1.1, we have

\[ p(z) < g(z), \quad z \in U, \]

and this result is sharp.

**Theorem 2.2** Let \( g \) be a convex function, \( g(0) = 0 \) and let \( h \) be the function \( h(z) = g(z) + \lambda zg(z), \ z \in U. \)

If \( f \in \Lambda \), \( \delta > 0 \), and satisfies the differential subordination

\[ \frac{(D_z^+H_m^t[\alpha_i]f(z))'}{z} < h(z), \quad z \in U, \]

then

\[ \frac{(D_z^+H_m^t[\alpha_i]f(z))'}{z} < g(z), \quad z \in U \]

and this result is sharp.

**Proof.** Consider \( p(z) = \frac{(D_z^+H_m^t[\alpha_i]f(z))'}{z} \), \( z \in U \).

Differentiating we obtain \( \frac{(D_z^+H_m^t[\alpha_i]f(z))'}{z} = p(z) + \frac{1}{\delta}zp'(z), \ z \in U. \)

Then (2.2) becomes

\[ p(z) + \frac{1}{\delta}zp'(z) < h(z) = g(z) + \lambda zg(z), \quad z \in U. \]

By using Lemma 1.1, we have

\[ p(z) < g(z), \ z \in U, \]

and this result is sharp.

**Theorem 2.3** Let \( g \) be a convex function such that \( g(0) = \frac{1}{1+\lambda} \) and let \( h \) be the function \( h(z) = g(z) + zg(z), \ z \in U. \)

If \( f \in \Lambda \) and the differential subordination

\[
\begin{aligned}
\alpha_i^2 \left( D_z^+H_m^t[\alpha_i+1]f(z) \right)' - \alpha_i (\alpha_i+1) D_z^+H_m^t[\alpha_i+2]f(z) \\
+ \frac{\alpha_i D_z^+H_m^t[\alpha_i+1]f(z) - \alpha_i(1+\lambda) D_z^+H_m^t[\alpha_i]f(z)}{\left[ \alpha_i D_z^+H_m^t[\alpha_i+1]f(z) - \alpha_i(1+\lambda) D_z^+H_m^t[\alpha_i]f(z) \right]^2} \\
+ 2 \alpha_i D_z^+H_m^t[\alpha_i]f(z) \cdot D_z^+H_m^t[\alpha_i+2]f(z) \\
- \left( \alpha_i(1+\lambda) D_z^+H_m^t[\alpha_i]f(z) \right)^2 < h(z), \quad z \in U.
\end{aligned}
\]

(2.3)
holds, then
\[
\frac{D_z^{-\lambda}H_m^\alpha [\alpha] f(z)}{z (D_z^{-\lambda}H_m^\alpha [\alpha] f(z))} < g(z), \quad z \in U.
\]
This result is sharp.

**Proof.** Let
\[
p(z) = \frac{D_z^{-\lambda}H_m^\alpha [\alpha] f(z)}{z (D_z^{-\lambda}H_m^\alpha [\alpha] f(z))}.
\]

Differentiating, we obtain
\[
1 - \frac{D_z^{-\lambda}H_m^\alpha [\alpha] f(z) \cdot (D_z^{-\lambda}H_m^\alpha [\alpha] f(z))'}{\left[(D_z^{-\lambda}H_m^\alpha [\alpha] f(z))\right]^2} = p(z) + zp'(z),
\]
\[z \in U.
\]

After a short calculation, using relation (1.6) we obtain
\[
1 - \frac{D_z^{-\lambda}H_m^\alpha [\alpha] f(z) \cdot (D_z^{-\lambda}H_m^\alpha [\alpha] f(z))'}{\left[(D_z^{-\lambda}H_m^\alpha [\alpha] f(z))\right]^2} = \frac{\alpha_1 (\alpha_1 + 1) f(z) - \alpha_1 (\alpha_1 + 1) D_z^{-\lambda}H_m^\alpha [\alpha] f(z) \cdot D_z^{-\lambda}H_m^\alpha [\alpha + 2] f(z)}{\alpha_1 D_z^{-\lambda}H_m^\alpha [\alpha + 1] f(z) - \alpha_1 (\alpha_1 + 1) D_z^{-\lambda}H_m^\alpha [\alpha] f(z)} - \frac{2 \alpha_1 D_z^{-\lambda}H_m^\alpha [\alpha] f(z) \cdot D_z^{-\lambda}H_m^\alpha [\alpha + 1] f(z) - \alpha_1 (\alpha_1 + 1) D_z^{-\lambda}H_m^\alpha [\alpha] f(z)}{\alpha_1 D_z^{-\lambda}H_m^\alpha [\alpha + 1] f(z) - \alpha_1 (\alpha_1 + 1) D_z^{-\lambda}H_m^\alpha [\alpha] f(z)}.
\]

Using the notation in (2.3), the differential subordination becomes
\[
p(z) + zp'(z) < h(z) = g(z) + zg'(z).
\]

By using Lemma 1.1, we have
\[
p(z) < g(z), \quad z \in U, \text{ i.e. } \frac{D_z^{-\lambda}H_m^\alpha [\alpha] f(z)}{z (D_z^{-\lambda}H_m^\alpha [\alpha] f(z))} < g(z), \quad z \in U,
\]
and this result is sharp.

**Theorem 2.4** Let \( g \) be a convex function such that \( g(0) = 0 \) and let \( h \) be the function
\[
h(z) = g(z) + \frac{\lambda}{\alpha_1 - \lambda} zg'(z), \quad z \in U.
\]

If \( f \in A \) and the differential subordination
\[
\frac{\alpha_1 (\alpha_1 + 1) D_z^{-\lambda}H_m^\alpha [\alpha + 2] f(z)}{\alpha_1 - \lambda} \frac{D_z^{-\lambda}H_m^\alpha [\alpha] f(z)}{z} < h(z), \quad z \in U
\]
holds, then
\[
\left(D_z^{-\lambda}H_m^\alpha [\alpha] f(z)\right) < g(z), \quad z \in U.
\]
This result is sharp.

**Proof.** Let
\[
p(z) = \left(D_z^{-\lambda}H_m^\alpha [\alpha] f(z)\right)\]

Differentiating and using relation (1.6), we obtain
\[
\frac{\alpha_1 (\alpha_1 + 1) D_z^{-\lambda}H_m^\alpha [\alpha + 2] f(z)}{\alpha_1 - \lambda} \frac{D_z^{-\lambda}H_m^\alpha [\alpha + 1] f(z)}{z} - \frac{\alpha_1 D_z^{-\lambda}H_m^\alpha [\alpha + 1] f(z)}{\alpha_1 - \lambda} = p(z) + \frac{1}{\alpha_1 - \lambda} zp'(z).
\]

Using the notation in (2.5), the differential subordination becomes
\[
p(z) + \frac{1}{\alpha_1 - \lambda} zp'(z) < h(z) = g(z) + \frac{\lambda}{\alpha_1 - \lambda} zg'(z).
\]

By using Lemma 1.1, we have
\[
p(z) < g(z), \quad z \in U, \text{ i.e. } \left(D_z^{-\lambda}H_m^\alpha [\alpha] f(z)\right) < g(z), \quad z \in U,
\]
and this result is sharp.

**REFERENCES**


