A Study of the Logistic Exponentiated-Exponential Distribution and Its Applications

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Abstract: The logistic-X (LX) family of distributions based on the logistic random variable was formulated recently by Tahir *et al.* [1]. We study a new special model of this family called the logistic exponentiated-exponential (LEE) distribution. Its density function can be symmetric, left-skewed, right-skewed, and reversed-J shaped, and its hazard rate can be decreasing and upside-down bathtub shapes. We provide a useful power series for its quantile function and a mixture representation for its density function. The parameters of the LEE model are estimated by maximum likelihood. Three Ozone data sets are modeled to illustrate the applicability of the new model.

Keywords: Exponentiated-exponential distribution, Logistic distribution, Logistic-X family, T-X family, Ozone data.

1. INTRODUCTION

The logistic distribution is a popular continuous model, and it is a strong competitor to the normal distribution since it has explicit formulas for the cumulative distribution function (cdf) and quantile function (qf) [2]. Both models are symmetric and bell-shaped on the support $\mathbb{R}$, but the logistic distribution has a heavier tail than the normal one. The logistic distribution has several applications in reliability and survival analysis, and it is useful for modeling growth phenomenons such as childhood cancer; respiratory disease prevalence due to smoking and age; geological issues; growth of human population; hemolytic uremic syndrome data analysis; physicochemical phenomenon; pneumoconiosis in coal miners, phycological tissues and study of diseases [3]; among others.

There has always been an interest for the researchers in defining and developing new distributions and generated families of univariate and bivariate distributions by introducing additional shape parameters to the baseline model.

Gupta and Kundu [4-12], in series of papers, introduced and studied the exponentiated-G family of distribution using Lehman's (1953) Alternative-I. Marshall-Olkin [13] proposed new method of adding parameter to the existing distribution. MO-Weibull (MOW) distribution was further studied [14-19]. MO-logistic-exponential by Mansoor *et al.* [20]. Some other well-known generators [1, 21], generalized raised cosine distribution by Ahsanullah *et al.* [22], beta-G [23,24], Kumaraswamy-G (Kw-G) by Cordeiro and de Castro [25], McDonald-G (Mc-G) [26,27] introduced Weibull power series distribution, Nadarajah and Kotz [28] developed beta-exponential distribution, Murthy et al. [29] gave a comprehensive detail on Weibull distribution and its extensions, gamma-G type 1 [30,31], gamma-G type 2 by Ristic and Balakrishnan [32], odd-gamma-G type 3 by Torabi and Montazari [33], logistic-G [34,35] introduced extended gamma Weibull family, odd exponentiated generalized by Cordeiro *et al.* [36], transformed-transformer (T-X) (Weibull-X and gamma-X) by Alzaatreh *et al.* [37], exponentiated T-X by Alzaghal *et al.* [38], odd Weibull-G by Bourguignon *et al.* [39], exponentiated half-logistic by Cordeiro *et al.* [40], T-X{Y}-quantile based approach by Aljarrah *et al.* [41] and T-R{Y} by Alzaatreh *et al.* [42], Poisson -X family by Tahir *et al.* [43], T-Lomax family by Mansoor *et al.* [44], Poisson Weibull-X by Mansoor *et al*. [45] and Lindley negative-binomial family by Mansoor *et al.* [46].

Let $r(t)$ be the pdf of a random variable $T \in [a,b]$ for $-\infty \le a < b < \infty$ and let $F(x)$ be the cdf of a random variable X such that the link function $W(\cdot):[0,1] \to [a,b]$ satisfies the two conditions: (i) $W(\cdot)$ is differentiable and monotonically non-decreasing, and (ii) $W(x) \to a$ as $x \to -\infty$ and $W(x) \to b$ as $x \to \infty$.

A random variable T has the one-parameter logistic distribution with shape parameter $\lambda > 0$, if its cdf and probability density function (pdf) (for $t \in \mathrm{R}$) are

$$R(t;\lambda) = (1+\mathrm{e}^{-\lambda t})^{-1} \quad \text{and} \quad r(t;\lambda) = \lambda \mathrm{e}^{-\lambda t}(1+\mathrm{e}^{-\lambda t})^{-2}, \tag{1}$$

respectively. A random variable having the logistic density in (1) is denoted by T : Logistic(λ). The survival function (sf) and hazard rate function (hrf) are

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$S(t;\lambda)=(1+e^{\lambda t})^{-1}$ and $\tau(t;\lambda)=\lambda(1+e^{-\lambda t})^{-1}$, respectively.

Numerous extended forms of distributions have been extensively used over the past decades for providing a better fit to real data in areas such as environmental and medical sciences, biological studies, demography, economics, actuarial, finance, insurance, and engineering. However, in many applied areas, several methods for generating new families will continue to be explored.

Let $G(x;\xi)$ and $\overline{G}(x;\xi)=1-G(x;\xi)$ be the baseline cdf and sf depending on a parameter vector ξ. Alzaatreh *et al.* [37] defined the *T-X family* by

$$F(x)=\int_a^{W[G(x)]} r(t)dt, \tag{2}$$

where $W[G(x)]$ satisfies the above conditions. The pdf corresponding to (2) becomes

$$f(x)=\frac{dW[G(x)]}{dx}r\{W[G(x)]\}. \tag{3}$$

By replacing $W[G(x)]$ by $log\left\{-log\left[\overline{G}(x;\xi)\right]\right\}$ and $r(t)$ in equation (2) by $r(t;\lambda)$ given by (1), Tahir *et al.* [1] defined the cdf and pdf of the *Logistic-X (LX) family* by

$$F(x;\lambda,\xi)=\left[1+\left\{-log[\overline{G}(x;\xi)]\right\}^{-\lambda}\right]^{-1} \tag{4}$$

and

$$f(x;\lambda,\xi)=\frac{\lambda g(x;\xi)}{\overline{G}(x;\xi)}\left\{-log\left[\overline{G}(x;\xi)\right]\right\}^{-(\lambda+1)}\left[1+\left\{-log[\overline{G}(x;\xi)]\right\}^{-\lambda}\right]^{-2}, \tag{5}$$

respectively. Note that equations (4) and (5) can be rewritten as

$$F(x)=[1+H_g(x)^{-\lambda}]^{-1}$$

and

$$f(x)=\lambda h_g(x)H_g(x)^{-(\lambda+1)}\left[1+H_g(x)^{-\lambda}\right]^{-2},$$

where $H_g(x)$ and $h_g(x)$ are the hazard and cumulative hazard functions corresponding to the pdf $g(x)$, respectively.

The generated family (5) allows us to extend well-known distributions and at the same time develop more realistic statistical models in a great variety of applications. The paper is unfolded as follows. In Section 2, we propose the *logistic exponentiated-exponential* ("LEE") distribution. In Section 3, its main structural properties are addressed. A useful representation for the LEE pdf is given in Section 4. In Section 5, the parameters of the LEE distribution are estimated by the method of maximum likelihood, and three real Ozone data sets are used to show the applicability of the LEE distribution. Section 6 offers some concluding remarks.

2. THE LEE DISTRIBUTION

Gupta and Kundu [47,48] pioneered and studied the two-parameter exponentiated-exponential (EE) distribution as an extension of the exponential distribution. The EE distribution is also known as the generalized exponential (GE) distribution in the literature. Since it is the most attractive generalization of the exponential distribution, the EE model has received increased attention, and several authors have studied its properties and proposed comparisons with other distributions.

A random variable Z has the EE distribution with scale parameter $\beta>0$ and shape parameter $\alpha>0$, if its cdf and pdf are given by (for $x>0$)

$$G(x)=(1-e^{-\beta x})^{\alpha} \quad \text{and} \quad g(x)=\alpha\beta e^{-\beta x}(1-e^{-\beta x})^{\alpha-1}, \tag{6}$$

respectively. We denote this distribution by EE(β,α). Now, using equation (4), we obtain the cdf of LEE distribution as (for $x>0$)

$$F(x)=F(x;\lambda,\beta,\alpha)=\left[1+\left\{-\log\left[1-(1-e^{-\beta x})^{\alpha}\right]\right\}^{-\lambda}\right]^{-1}. \tag{7}$$

The pdf corresponding to (7) is given by

$$f(x)=\frac{\lambda\alpha\beta e^{-\beta x}\left(1-e^{-\beta x}\right)^{\alpha-1}}{1-(1-e^{-\beta x})^{\alpha}}\left\{-\log\left[1-(1-e^{-\beta x})^{\alpha}\right]\right\}^{-\lambda-1}$$
$$\times\left[1+\left\{-\log\left[1-(1-e^{-\beta x})^{\alpha}\right]\right\}^{-\lambda}\right]^{-2}. \tag{8}$$

A random variable having the pdf (8) will be denoted by $X:\mathrm{LEE}(\lambda,\beta,\alpha)$. The survival and hazard rate functions of X are, respectively, $S(x)=1-F(x)$ and $h(x)=f(x)/S(x)$ where $f(x)$ and $F(x)$ are given in (8) and (7). We can write

$$S(x)=1-\left[1+\left\{-\log\left[1-(1-e^{-\beta x})^{\alpha}\right]\right\}^{-\lambda}\right]^{-1}, \tag{9}$$

$$h(x)=\frac{\lambda\alpha\beta e^{-\beta x}\left(1-e^{-\beta x}\right)^{\alpha-1}}{1-(1-e^{-\beta x})^{\alpha}}\left\{-\log\left[1-(1-e^{-\beta x})^{\alpha}\right]\right\}^{-\lambda-1}$$

$$\times\left[1+\left\{-\log\left[1-(1-e^{-\beta x})^{\alpha}\right]\right\}^{-\lambda}\right]^{-2}$$

$$\times\left\{1-\left[1+\left\{-\log\left[1-(1-e^{-\beta x})^{\alpha}\right]\right\}^{-\lambda}\right]^{-1}\right\}^{-1}$$

and the cumulative hazard function (chf)

$$H(x)=\log\left\{1-\left[1+\left\{-\log\left[1-(1-e^{-\beta x})^{\alpha}\right]\right\}^{-\lambda}\right]^{-1}\right\},$$

respectively.

2.1. Shapes of the Density and Hazard Rate Functions

The shapes of the density and hazard rate functions can be described analytically. The critical points of the LEE density are the roots of the equation:

$$\frac{(1-\alpha e^{-\beta x})}{(1-e^{-\beta x})}=\frac{\alpha e^{-\beta x}w^{1-1/\alpha}}{(1-w)}\left\{\frac{-(\lambda+1)}{\log(1-w)}-\frac{2\left[-\log(1-w)\right]^{-\lambda-1}}{\left[1+\left\{-\log(1-w)\right\}^{-\lambda}\right]}+1\right\},$$

where $w=w(x)=(1-e^{-\beta x})^{\alpha}$.

The critical points of the LEE hazard rate are obtained from

$$\beta w+\left(\frac{1}{\alpha}-1\right)w'=\frac{ww'}{w-1}\left\{\frac{1}{\log(1-w)}+\frac{2\lambda\left[-\log(1-w)\right]^{-\lambda-1}}{(1-w)\left\{1+\left[-\log(1-w)\right]^{-\lambda}\right\}}\right.$$
$$\left.+\frac{\lambda\left[-\log(1-w)\right]^{-\lambda-1}}{\left[1-\left\{1+\left[-\log(1-w)\right]^{-\lambda}\right\}^{-1}\right]}\right\}.$$

Using any numerical software, we can examine the last two equations to determine the local maximums and minimums, and inflexion points.

Figures **1** and **2** display some plots of the pdf and hrf of X for some parameter values. Figure **1** indicates that the LEE distribution can be right-skewed, left-skewed, and reversed J shapes. Also when $\alpha\geq 1$ and $\lambda<1$, the LEE distribution is right-skewed. Note that the skewness increases when $\alpha\lambda>1$. The plots in Figure **2** show that the LEE hrf possesses various shapes, including decreasing and upside-down bathtub shapes.

3. SOME PROPERTIES

In this section, we study some general properties for the LEE distribution, including quantile function, moments, and Shannon entropy. The formulae derived throughout the paper can be easily handled in symbolic computation software like Maple, Mathematica, and Matlab.

Lemma 3.1 If $Y:Logistic(\lambda)$ then $X=\log\left[1-(1-e^{-e^{Y}})^{\frac{1}{\alpha}}\right]^{-\frac{1}{\beta}}$ follows the $LEE(\lambda,\alpha,\beta)$ distribution.

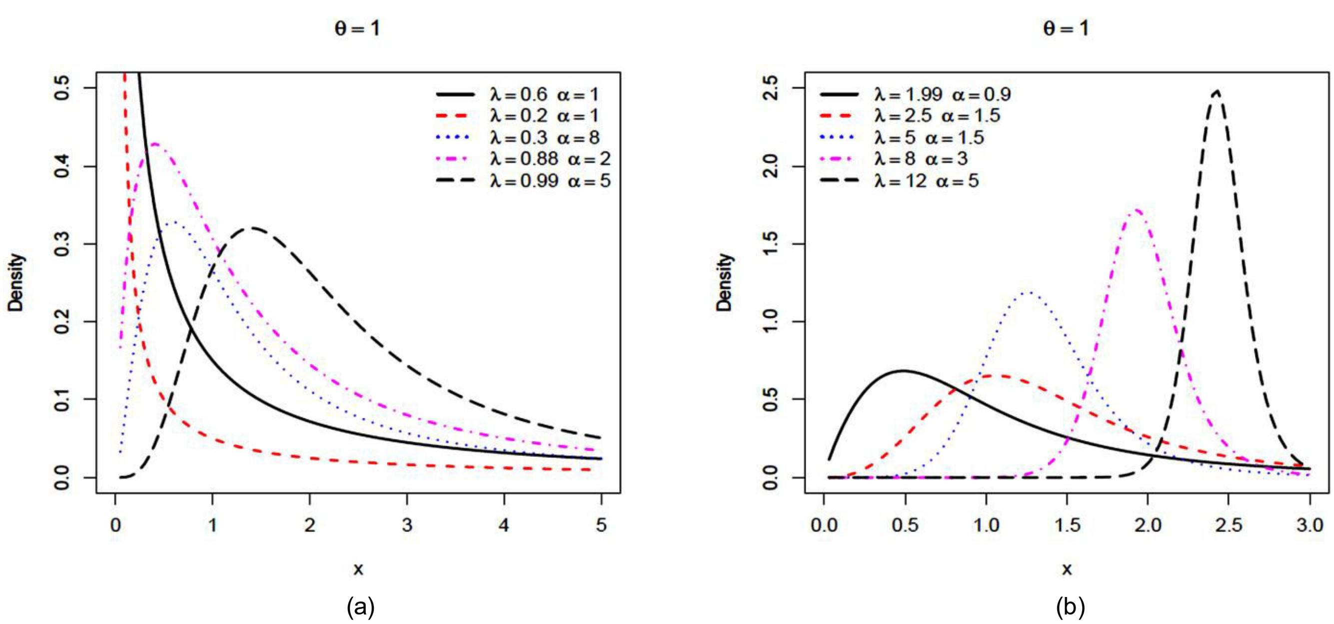
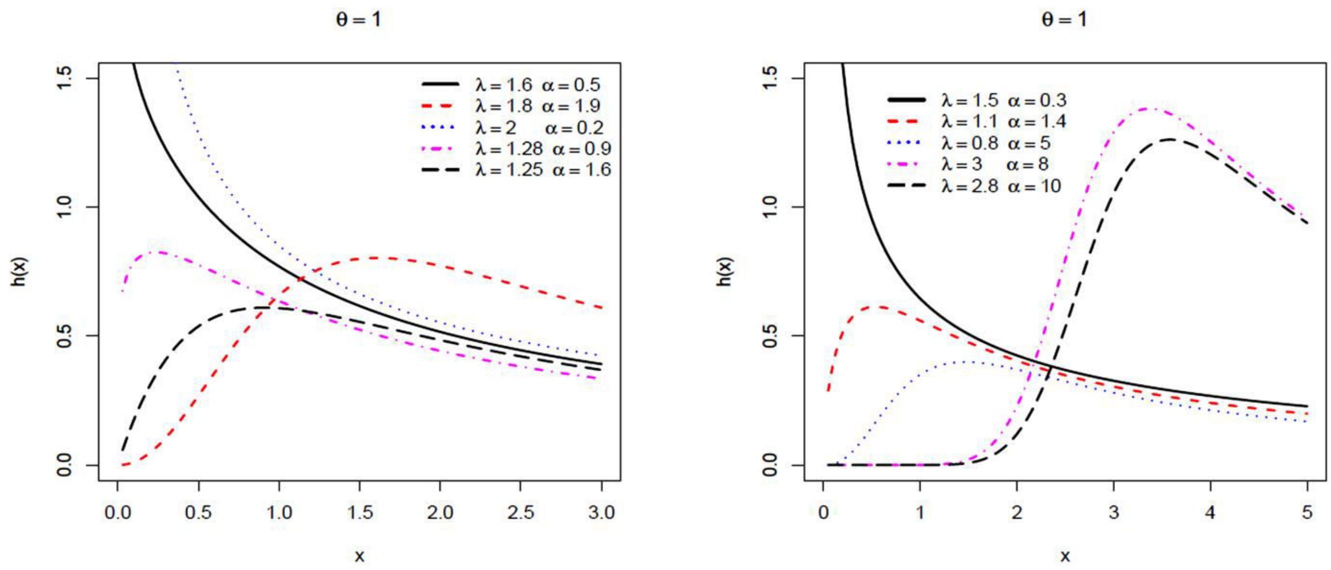


Figure 1: Plots of the LEE density for some values of λ and α.



(a) **(b)**

Figure 2: Plots of the LEE hazard rate for some values of λ and α.

Remark 3.1 The qf of X is obtained by inverting (7) as (for $u \in (0,1)$)

$$x = Q(u) = -\beta^{-1}\log\{1-[1-\mathrm{e}^{\left(\frac{1-u}{u}\right)^{-\frac{1}{\lambda}}}]^{\frac{1}{\alpha}}\}. \tag{10}$$

If U has a uniform distribution in $(0,1)$, then $X = Q(U)$ has the $\mathrm{L}EE(\lambda,\beta,\alpha)$ distribution.

Theorem 3.1 The k th moment of X is given by

$$\mu_k' = \mathrm{E}(X^k) = \sum_{l,m=0}^{\infty}\frac{(-1)^m b_l^{(k)}}{\beta^k}\binom{\frac{k+l}{\alpha}}{m}$$

$$\times\left[1+\sum_{p=1}^{\infty}\frac{(-1)^p m^p}{\Gamma(p+1)}\Gamma(1-\tfrac{p}{\lambda})\Gamma(1+\tfrac{p}{\lambda})\right]. \tag{11}$$

where (for $k = 1,2,\ldots$ and $l = 0,1,2,\ldots$)

$$b_l^{(k)} = \sum_{j=0}^{l} b_j^{(k-1)} b_{l-j} \quad \text{and} \quad b_j = b_j^{(0)} = \frac{1}{j+1}.$$

Proof. Based on Lemma 3.1,

$$\mathrm{E}(X^k) = (-\beta)^{-k}\int_{-\infty}^{\infty}\left\{\log\left[1-(1-\mathrm{e}^{-\mathrm{e}^x})^{\frac{1}{\alpha}}\right]\right\}^k \frac{\lambda \mathrm{e}^{-\lambda x}}{\left(1+\mathrm{e}^{-\lambda x}\right)^2}dx. \tag{12}$$

For $k = 1,2,\ldots$ and $z \in (0,1)$, the power series holds

$$[\log(1-z)]^k = (-1)^k z^k \sum_{l=0}^{\infty} b_l^{(s)} z^l,$$

where $b_l^{(k)} = \sum_{j=0}^{l} b_j^{(k-1)} b_{l-j}$ (for $l = 0,1,2,\ldots$) and $b_j = b_j^{(0)} = 1/(j+1)$.

We can write

$$\{\log\left[1-(1-\mathrm{e}^{-\mathrm{e}^x})^{\frac{1}{\alpha}}\right]\}^k = (-1)^k\sum_{l=0}^{\infty} b_l^{(k)}(1-\mathrm{e}^{-\mathrm{e}^x})^{\frac{k+l}{\alpha}}.$$

By using the generalized binomial expansion in the last term, we have

$$\{\log\left[1-(1-\mathrm{e}^{-\mathrm{e}^x})^{\frac{1}{\alpha}}\right]\}^k = (-1)^k\sum_{l=0}^{\infty} b_l^{(k)}\sum_{m=0}^{\infty}(-1)^m\binom{\frac{k+l}{\alpha}}{m}\mathrm{e}^{-m\mathrm{e}^x}.$$

By expanding the exponential function in power series gives

$$\{\log\left[1-(1-\mathrm{e}^{-\mathrm{e}^x})^{\frac{1}{\alpha}}\right]\}^k = (-1)^k\sum_{l=m=0}^{\infty}(-1)^m b_l^{(k)}\binom{\frac{k+l}{\alpha}}{m}$$

$$\times\left[1+\sum_{p=1}^{\infty}\frac{(-1)^p m^p \mathrm{e}^{px}}{\Gamma(p+1)}\right]. \tag{13}$$

Equation (**11**) follows by substituting (**13**) in equation (12) and noting that

$$\int_{-\infty}^{\infty}\frac{\lambda e^{px}e^{-\lambda x}}{\left(1+e^{-\lambda x}\right)^2}dx=\Gamma(1-\tfrac{p}{\lambda})\Gamma(1+\tfrac{p}{\lambda}).$$

Theorem 3.2 The kth incomplete moment of X can be expressed (for $y>0$) as

$$m_k(y)=\sum_{l,m=0}^{\infty}\frac{(-1)^m b_l^{(k)}}{\beta^k}\binom{\frac{k+l}{\alpha}}{m}$$

$$\times\left[1+\sum_{p=1}^{\infty}\frac{(-1)^p m^p}{\Gamma(p+1)}\frac{y^{-1-\frac{p}{\lambda}}\,{}_2F_1(2,1+\frac{p}{\lambda};2+\frac{p}{\lambda};-y^{-1})}{\left(1+\frac{p}{\lambda}\right)}\right]. \quad (14)$$

Proof. The kth incomplete moment of X follows from the following result. This result can be found in [49].

$$\int_y^{\infty}\frac{x^{-\frac{p}{\lambda}}}{(1+x)^2}dx=\frac{y^{-1-\frac{p}{\lambda}}\,{}_2F_1(2,1+\frac{p}{\lambda};2+\frac{p}{\lambda};-y^{-1})}{\left(1+\frac{p}{\lambda}\right)},y>0,$$

where ${}_2F_1$ is the hypergeometric function given by

$${}_2F_1(a,b;c;x)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{j=0}^{\infty}\frac{\Gamma(a+j)\Gamma(b+j)}{\Gamma(c+j)}\frac{x^j}{j!}.$$

The main application of the first incomplete moment refers to the Bonferroni and Lorenz curves. These curves are very useful in economics, reliability, demography, insurance, and medicine. For a given probability π, they are defined by $B(\pi)=m_1(q)/(\pi\mu_1^{'})$ and $L(\pi)=m_1(q)/\mu_1^{'}$, respectively, where $m_1(q)$ is obtained from (**14**) with $k=1$, and $q=Q(\pi)$ is determined from (20) given in Section 4.1.

The amount of scattering in a population is measured to some extent by the totality of deviations from the mean and median defined by $\delta_1=\int_0^{\infty}|x-\mu_1^{'}|f(x)dx$ and $\delta_2(x)=\int_0^{\infty}|x-M|f(x)dx$, respectively, where $\mu_1^{'}=E(X)$ is the mean and $M=Q(0.5)$ is the median. These measures can be expressed as $\delta_1=2\mu_1^{'}F(\mu_1^{'})-2m_1(\mu_1^{'})$ and $\delta_2=\mu_1^{'}-2m_1(M)$, where $F(\mu_1^{'})$ comes from (7).

Further applications of the first incomplete moment are related to the mean residual life and mean waiting time given by $\nu(t)=[1-m_1(t)]/S(t)-t$ and $\mu(t)=t-[m_1(t)/F(t)]$, respectively, where $F(t)$ and $S(t)=1-F(t)$ are obtained from (7).

Theorem 3.3 The Shannon's entropy of X is given by

$$\eta_x=\log(\alpha\beta)-\sum_{k=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^m}{k}\binom{k/\alpha}{m}\left[1+\sum_{p=1}^{\infty}\frac{(-1)^p m^p}{\Gamma(p+1)}\Gamma(1-\tfrac{p}{\lambda})\Gamma(1+\tfrac{p}{\lambda})\right]$$

$$-\left(1-\tfrac{1}{\alpha}\right)\sum_{k=1}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^j k^{j-1}}{j!}\Gamma(1-\tfrac{j}{\lambda})\Gamma(1+\tfrac{j}{\lambda})-B\left(1-\tfrac{1}{\lambda},1+\tfrac{1}{\lambda}\right)$$

$$-log\lambda+2. \quad (15)$$

Proof. Based on Tahir *et al.* [1], the Shannon entropy of the LX family can be expressed as

$$\eta_x=E\left[log\left\{g\left[G^{-1}\left(1-e^{-e^T}\right)\right]\right\}\right]-B\left(1-\tfrac{1}{\lambda},1+\tfrac{1}{\lambda}\right)-log\lambda+2, \quad (16)$$

where T: Logistic(λ).

First, we obtain $E\left[log\left\{g\left[G^{-1}\left(1-e^{-e^T}\right)\right]\right\}\right]$, where $g(x)$ and $G(x)$ are the pdf and cdf of the EE distribution. Then, we have

$$\log\left\{g\left[G^{-1}\left(1-e^{-e^T}\right)\right]\right\}=\log(\alpha\beta)+\log\left\{1-\left(1-e^{-e^T}\right)^{1/\alpha}\right\}$$

$$+\left(1-\tfrac{1}{\alpha}\right)\log\left(1-e^{-e^T}\right). \quad (17)$$

We use the well-known power series

$$\log\left(1-e^{-e^T}\right)=-\sum_{k=1}^{\infty}k^{-1}e^{-ke^T}. \quad (18)$$

Based on the power series and generalized binomial expansion, we have

$$\log\left\{1-\left(1-e^{-e^T}\right)^{1/\alpha}\right\}=-\sum_{k=1}^{\infty}k^{-1}\sum_{m=0}^{\infty}(-1)^m\binom{k/\alpha}{m}e^{-me^T}.$$

From the last two equations, we can write

$$\mathrm{E}\left[\log\left\{g\left[G^{-1}\left(1-\mathrm{e}^{-\mathrm{e}^T}\right)\right]\right\}\right]=\log(\alpha\beta)$$

$$-\sum_{k=1}^{\infty}\sum_{m=0}^{\infty}k^{-1}(-1)^m\binom{k/\alpha}{m}\mathrm{E}\left[\mathrm{e}^{-m\mathrm{e}^T}\right]$$

$$-\left(1-\tfrac{1}{\alpha}\right)\sum_{k=1}^{\infty}k^{-1}\mathrm{E}\left(\mathrm{e}^{-k\mathrm{e}^T}\right). \tag{19}$$

Then, equation (**15**) follows by substituting (**19**) in (**17**) and noting that

$$\mathrm{E}\left(\mathrm{e}^{-m\mathrm{e}^T}\right)=\left[1+\sum_{p=1}^{\infty}\frac{(-1)^p m^p}{\Gamma(p+1)}\Gamma(1-\tfrac{p}{\lambda})\Gamma(1+\tfrac{p}{\lambda})\right]$$

and

$$\mathrm{E}\left(\mathrm{e}^{-k\mathrm{e}^T}\right)=\sum_{j=0}^{\infty}\frac{(-1)^j k^j}{j!}\Gamma(1-\tfrac{j}{\lambda})\Gamma(1+\tfrac{j}{\lambda}).$$

4. USEFUL REPRESENTATIONS

4.1. Quantile Power Series

Let $z=u^{-1}(1-u)$. The qf of X follows by inverting (7) and using (18)

$$Q(u)=-\beta^{-1}\sum_{i=1}^{\infty}\frac{1}{i}(1-\mathrm{e}^{z^{-\frac{1}{\lambda}}})^{i/\alpha}. \tag{20}$$

By expanding the binomial and then using the power series for the exponential function, we have

$$Q(u)=-\beta^{-1}\sum_{i=1}^{\infty}\frac{1}{i}\left[1+\sum_{j=1}^{\infty}\sum_{k,r=0}^{\infty}\frac{(-1)^{j+r}j^k}{k!}\binom{i/\alpha}{j}\binom{-k/\lambda}{r}u^{r+k/\lambda}\right].$$

For $r\geq 0$ and $k\geq 0$, we define the quantities:

$$w_{r,k}=\beta^{-1}\sum_{i,j=1}^{\infty}\frac{(-1)^{j+r+1}j^k}{ik!}\binom{i/\alpha}{j}\binom{-k/\lambda}{r}$$

when $r+k\geq 1$ and $w_{0,0}=\beta^{-1}\sum_{i,j=1}^{\infty}\left[-1+\frac{(-1)^{j+1}}{i}\binom{i/\alpha}{j}\right]$.

Then, we can rewrite $Q(u)$ as

$$Q(u)=\sum_{r,k=1}^{\infty}w_{r,k}\,u^{r+k/\lambda}.$$

The following power series holds for any real non-integer power and $u\in(0,1)$

$$u^{r+k/\lambda}=\sum_{i=0}^{\infty}s_i(r+k/\lambda)u^i,$$

where $s_i=s_i(r+k/\lambda)=\sum_{j=0}^{\infty}(-1)^{i+j}\binom{r+k/\lambda}{j}\binom{j}{i}$. Then, we can rewrite $Q(u)$ as

$$Q(u)=\sum_{i=0}^{\infty}t_i u^i, \tag{21}$$

where $t_i=t_i(\lambda)=\sum_{r,k=1}^{\infty}w_{r,k}\,s_i(r+k/\lambda)$.

Equation (21) is the main result of this section since it allows to obtain some mathematical quantities for the LEE distribution. Let $W(\cdot)$ be any integrable function in the positive real line. We can write

$$\int_0^{\infty}W(x)f(x)dx=\int_0^1 W\left(\sum_{i=0}^{\infty}t_i u^i\right)du.$$

So, several mathematical quantities of X can be derived with integrals over $(0,1)$.

4.2. Mixture representation

From (7) we have

$$F(x)=\frac{1}{1+\left\{-\log\left[1-(1-\mathrm{e}^{-\beta x})^{\alpha}\right]\right\}^{-\lambda}}. \tag{22}$$

Let $w=1+[-\log(1-x)]^a$. For $a<0$ and $0<x<1$, the Mathematica software gives the power series for w

$$w=1+[1+\frac{a}{2}x+\frac{1}{24}(3a^2+5a)x^2+\frac{1}{48}(a^3+5a^2+6a)x^3$$

$$+\frac{1}{5760}(15a^4+150a^3+485a^2+502a)x^4$$

$$+\frac{1}{11520}(3a^5+50a^4+305a^3+802a^2+760a)x^5]x^a+O(x^{a+6}).$$

Using the last equation, we can write

$$1+\left\{-\log\left[1-(1-\mathrm{e}^{-\beta x})^{\alpha}\right]\right\}^{-\lambda}=1+(1-\mathrm{e}^{-\beta x})^{-\lambda\alpha}\sum_{k=0}^{\infty}p_k(1-\mathrm{e}^{-\beta x})^{k\alpha}, \quad (23)$$

where the p_k are given by $p_0=1$, $p_1=-\lambda/2$, $p_2=(3\lambda^2-5\lambda)/24$, $p_3=(-\lambda^3+5\lambda^2-6\lambda)/48$, $p_4=(15\lambda^4-150\lambda^3+485\lambda^2-502\lambda)/5760$, etc.

Further, the following expansion holds for any $\lambda>0$ real non-integer

$$\left[(1-\mathrm{e}^{-\beta x})^{\alpha}\right]^{\lambda}=\sum_{k=0}^{\infty}q_k(1-\mathrm{e}^{-\beta x})^{k\alpha}, \quad (24)$$

where $q_k=q_k(\lambda)=\sum_{j=0}^{\infty}(-1)^{k+j}\binom{\lambda}{j}\binom{j}{k}$.

Combining (23) and (24), equation (22) becomes

$$F(x;\lambda,\beta,\alpha)=\frac{\sum_{k=0}^{\infty}q_k(1-\mathrm{e}^{-\beta x})^{k\alpha}}{\sum_{k=0}^{\infty}q_k(1-\mathrm{e}^{-\beta x})^{k\alpha}+\sum_{k=0}^{\infty}p_k(1-\mathrm{e}^{-\beta x})^{k\alpha}}$$

$$=\frac{\sum_{k=0}^{\infty}q_k(1-\mathrm{e}^{-\beta x})^{k\alpha}}{\sum_{k=0}^{\infty}v_k(1-\mathrm{e}^{-\beta x})^{k\alpha}},$$

where $v_k=p_k+q_k$ for $k=0,1,\ldots$

The quotient of the two power series in the last equation reduces to

$$F(x)=\sum_{k=0}^{\infty}c_k(1-\mathrm{e}^{-\beta x})^{k\alpha}, \quad (25)$$

where $c_0=q_0/v_0$ and the coefficients c_k's (for $k\geq 1$) are determined from the recurrence equation

$$c_k=\frac{1}{v_0}\left(q_k+\frac{1}{v_0}\sum_{r=1}^{k}v_r\,c_{k-r}\right).$$

By differentiating (25), the pdf of X can be rewritten as a mixture of EE density functions

$$f(x)=\sum_{k=0}^{\infty}c_{k+1}\,\pi_{(k+1)\alpha,\beta}(x), \quad (26)$$

where $\pi_{(k+1)\alpha,\beta}(x)=(k+1)\alpha\beta\mathrm{e}^{-\beta x}(1-\mathrm{e}^{-\beta x})^{(k+1)\alpha-1}$ (for $k\geq 0$) is the EE pdf with the common scale parameter β, and the power parameter $(k+1)\alpha$. Equation (26) is the main result of this section. The mathematical properties of the LEE model can then be derived from those of the EE density function, which have been explored exhaustively. See, for example [47,48].

A simple application of (26) can be given to the moment generating function (mgf) of X, say $M(t)$. It can be immediately derived from (26) and the well-known result for the mgf of the EE distribution. We obtain (for $t<\lambda$)

$$M(t)=\Gamma(1-t/\lambda)\sum_{k=0}^{\infty}\frac{c_{k+1}\,\Gamma((k+1)\alpha+1)}{\Gamma((k+1)\alpha+1-t/\lambda)}.$$

5. ESTIMATION AND APPLICATIONS

Here, we consider the estimation of the unknown parameters of the LEE distribution by the maximum likelihood method. The maximum likelihood estimates (MLEs) enjoy desirable properties that can be used when constructing confidence intervals and deliver simple approximations that work well in finite samples. The resulting approximation for the MLEs in distribution theory is easily handled either analytically or numerically. Let $x_1,\ldots,x_n$ be a sample of size n from the LEE distribution given by (**8**). The log-likelihood function for the vector of parameters $\mathbf{\Theta}=(\lambda,\beta,\alpha)^{\mathrm{T}}$ can be expressed as

$$\ell=n\log(\lambda\alpha\beta)-\beta\sum_{i=1}^{n}x_i+(\alpha-1)\sum_{i=1}^{n}\left(1-\mathrm{e}^{-\beta x_i}\right)$$

$$-\sum_{i=1}^{n}\left[1-\left(1-\mathrm{e}^{-\beta x_i}\right)^{\alpha}\right]-(\lambda+1)\sum_{i=1}^{n}\left\{-\log\left[1-\left(1-\mathrm{e}^{-\beta x_i}\right)^{\alpha}\right]\right\}$$

$$-2\sum_{i=1}^{n}[1+\left\{-\log\left[1-\left(1-\mathrm{e}^{-\beta x_i}\right)^{\alpha}\right]\right\}^{-\lambda}].$$

Maximization of ℓ can be performed by using well-established routines like NLM or OPTIMIZE in the R statistical package, the NLMIXED procedure in SAS or the MaxBFGS in the Ox program.

The components of the score vector $U(\mathbf{\Theta})$ are

$$U_{\lambda}=\frac{n}{\lambda}+\sum_{i=1}^{n}\log\left[1-\left(1-\mathrm{e}^{-\beta x_i}\right)^{\alpha}\right]+2\sum_{i=1}^{n}\left\{-\log\left[1-\left(1-\mathrm{e}^{-\beta x_i}\right)^{\alpha}\right]\right\}^{-\lambda}$$

$$\times \log\left\{-\log\left[1-\left(1-e^{-\beta x_i}\right)^{\alpha}\right]\right\},$$

$$U_{\beta}=\frac{n}{\beta}-\sum_{i=1}^{n}x_i+(\alpha-1)\sum_{i=1}^{n}x_i e^{-\beta x_i}+\alpha\sum_{i=1}^{n}x_i e^{-\beta x_i}\left(1-e^{-\beta x_i}\right)^{\alpha-1}$$

$$-\alpha(\lambda+1)\left[\frac{x_i e^{-\beta x_i}\left(1-e^{-\beta x_i}\right)^{\alpha-1}}{\left[1-\left(1-e^{-\beta x_i}\right)^{\alpha}\right]}\right]$$

$$+2\alpha\lambda\left[\frac{x_i e^{-\beta x_i}\left(1-e^{-\beta x_i}\right)^{\alpha-1}\left\{-\log\left[1-\left(1-e^{-\beta x_i}\right)^{\alpha}\right]\right\}^{-(\lambda+1)}}{\left[1-\left(1-e^{-\beta x_i}\right)^{\alpha}\right]}\right],$$

$$U_{\alpha}=\frac{n}{\alpha}+\sum_{i=1}^{n}\left(1-e^{-\beta x_i}\right)+\sum_{i=1}^{n}\left(1-e^{-\beta x_i}\right)^{\alpha}\log\left(1-e^{-\beta x_i}\right)$$

$$-(\lambda+1)\sum_{i=1}^{n}\left[\frac{\left(1-e^{-\beta x_i}\right)^{\alpha}\log\left(1-e^{-\beta x_i}\right)}{\left[1-\left(1-e^{-\beta x_i}\right)^{\alpha}\right]}\right].$$

Setting U_{λ}, U_{β} and U_{α} equal to zero and solving these equations simultaneously yields the MLEs $\hat{\Theta}=(\hat{\lambda},\hat{\beta},\hat{\alpha})$ ú.

5.1. Applications to Real-Life Data

In this section, we use three real data sets and fit the LEE model. All the calculations were performed by R Development Core Team [50] software.

Data Set 1:

The first Ozone data set is taken from Nadarajah [51]. This data represents measurements of daily ozone concentration (ppb) on 111 days from May to September 1973 in New York. The data are: 41, 36, 12, 18, 28, 23, 19, 8, 7, 16, 11, 14, 18, 14, 34, 6, 30, 11, 1, 11, 4, 32, 23, 45, 115, 37, 29, 71, 39, 23, 21, 37, 20, 12, 13, 135, 49, 32, 64, 40, 77, 97, 97, 85, 10, 27, 7, 48, 35, 61, 79, 63, 16, 80, 108, 20, 52, 82, 50, 64, 59, 39, 9, 16, 78, 35, 66, 122, 89, 110, 44, 28, 65, 22, 59, 23, 31, 44, 21, 9, 45, 168, 73, 76, 118, 84, 85, 96, 78, 73, 91, 47, 32, 20, 23, 21, 24, 44, 21, 28, 9, 13, 46, 18, 13, 24, 16, 13, 23, 36, 7, 14, 30, 14, 18, 20.

Data Set 2:

The second data set represents the average daily ozone values over 1987 summer at 20 Chicago monitoring stations on the website: www.image.ucar.edu/GSP/Software/Fields/Help/ozone.html. The data are: 59, 58, 90, 80, 50, 47, 81, 56, 55, 72, 62, 100, 97, 91, 80, 81, 76, 75, 85, 94, 80, 82, 74, 68, 60, 85, 34, 66, 65, 73, 63, 62, 36, 54, 42, 52, 64, 65, 60, 56, 64.

Data Set 3:

This data set records the level of atmospheric ozone concentration from eight daily meteorological measurements made in the Los Angeles basin in 1976. Although measurements were made every day that year, some observations were missing; here, we have the 330 complete cases. These data can be accessed using the following link: http://www-stat.stanford.edu/tibs/ElemStatLearn/datasets. The response, referred to as ozone, is actually the log of the daily maximum of the hourly-average ozone concentrations in Upland, California.

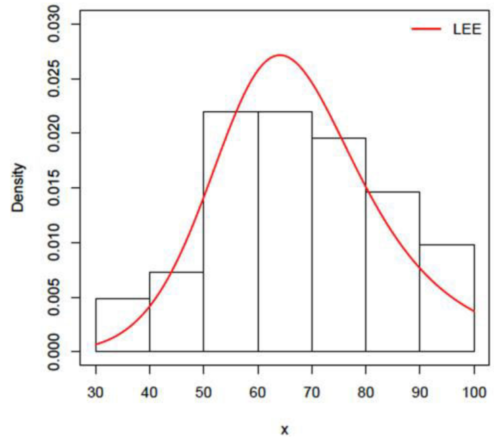
We fit the LEE to these data sets. We present the MLEs, their Standard Errors (SEs) in parentheses, the Akaike information criterion (AIC), the Kolmogrov-Smirnov (K-S) statistics and associated P-values in Table **1**. The K-S statistic and its P-value given in Table 1 indicate that LEE model provides an adequate fit. For a visual comparison, we provide the empirical and fitted pdf and cdf of the LLE model in Figure **3**. Clearly, the LEE model provides a closer fit to the data.

6. CONCLUDING REMARKS

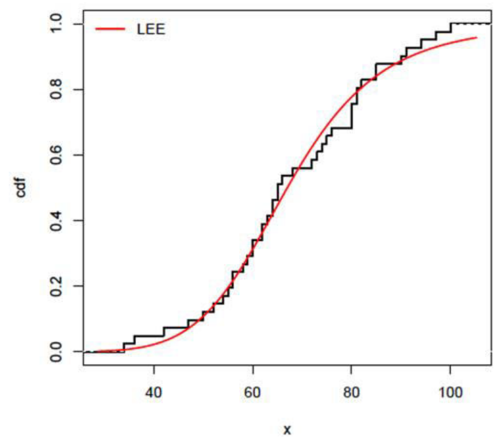
In this paper, we studied the *logistic-exponentiated exponential (LEE)* distribution which is a member of the *Logistic-X* (LX) family introduced by Tahir *et al.* [1]. We studied some mathematical properties of the LEE

Table 1: MLEs, their SEs (in Parentheses) and Goodness-of-Fit Measures for Three Data Sets

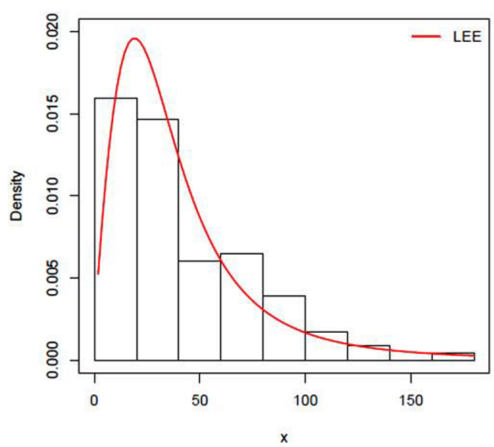
Data Set	α	λ	β	AIC	K-S	P-Value
data 1	0.4472(0.2014)	3.3156(1.0519)	0.0139(0.0081)	1093.00	0.0703	0.614
data 2	1.5668(0.0933)	5.6540(1.9976)	0.0204(0.0090)	352.44	0.0942	0.860
data 3	0.3204(0.1205)	4.7111(1.3680)	0.02861(0.0177)	2225.91	0.0811	0.225



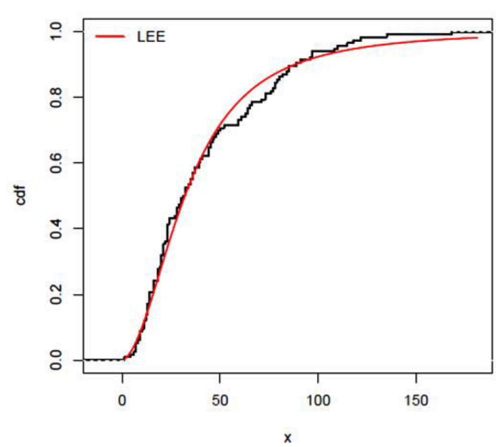
(a) Estimated pdfs for data set 1



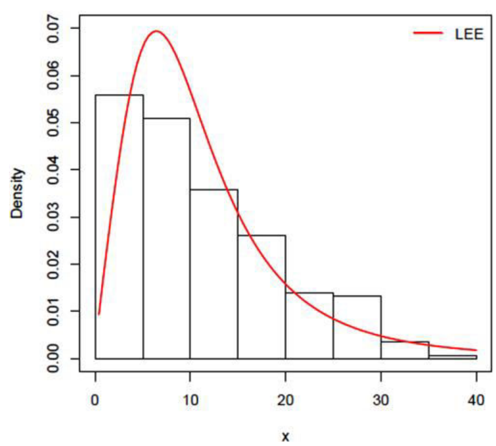
(b) Estimated cdfs for data set 1



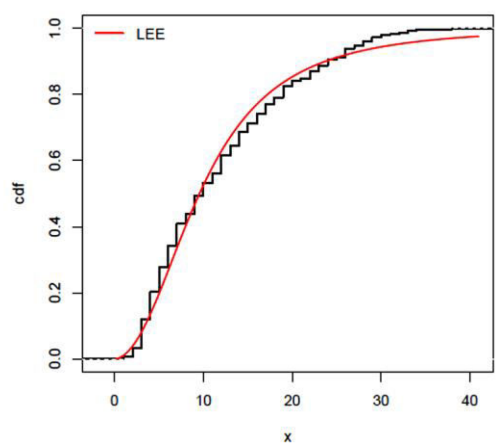
(a) Estimated pdfs for data set 2



(b) Estimated cdfs for data set 2



(a) Estimated pdfs for data set 3



(b) Estimated cdfs for data set 3

Figure 3: Plots of the estimated pdfs and cdfs of the LEE model for data sets 1, 2 and 3.

model, including explicit expressions for the quantile function, ordinary and incomplete moments, mean deviations, Shannon entropy, and generating function. The density function of the proposed distribution can be expressed in terms of exponentiated exponential densities. The maximum likelihood method is employed for estimating the model parameters. We fit the LEE distribution to three Ozone data sets to demonstrate its flexibility.

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REFERENCES

[1] Tahir MH, Cordeiro GM, Alzaatreh A, Mansoor M, Zubair M. The Logistic-X family of distributions and its applications. Communications in Statistics-Theory and Methods, 2016; 45: 7326-7349. https://doi.org/10.1080/03610926.2014.980516

[2] Johnson NL, Kotz S, Balakrishnan N. Continuous Univariate Distributions. 1995; Vol. 2: 2nd eds, Wiley, New York.

[3] Nadarajah S. Information matrix for logistic distributions. Mathematical and Computer Modelling 2004; 91: 689-697. https://doi.org/10.1016/j.mcm.2004.04.002

[4] Gupta RD, Kundu D. Generalized exponential distribution: An alternative to Gamma and Weibull distributions. Biometrical Journal 2001; 43: 117-130. https://doi.org/10.1002/1521-4036(200102)43:1<117::AID-BIMJ117>3.0.CO;2-R

[5] Gupta RD, Kundu D. Generalized exponential distribution: Different methods of estimations. Journal of Statistical Computation and Simulation 2001; 69: 315-337. https://doi.org/10.1080/00949650108812098

[6] Gupta RD, Kundu D. Discriminating between the Weibull and the GE distributions. Computational Statistics and Data Analysis 2002; 43: 179-196. https://doi.org/10.1016/S0167-9473(02)00206-2

[7] Gupta RD, Kundu D. Closeness of gamma and generalized exponential distributions. Communications in Statistics-Theory and Methods 2003; 32: 705-721. https://doi.org/10.1081/STA-120018824

[8] Gupta RD, Kundu D. Discriminating between the gamma and generalized exponential distributions. Journal of Statistical Computation and Simulation 2004; 74: 107-121. https://doi.org/10.1080/0094965031000114359

[9] Gupta RD, Kundu D. On comparison of the Fisher information of the Weibull and GE distributions. Journal of Statistical Planning and Inference 2006; 136: 3130-3144. https://doi.org/10.1016/j.jspi.2004.11.013

[10] Gupta RD, Kundu D. Generalized exponential distribution: Existing results and some recent developments.Journal of Statistical Planning and Inference 2007; 137: 3537-3547. https://doi.org/10.1016/j.jspi.2007.03.030

[11] Gupta RD, Kundu D. Generalized exponential distribution: Bayesian Inference. Computational Statistics and Data Analysis 2008; 52: 1873-1883. https://doi.org/10.1016/j.csda.2007.06.004

[12] Gupta RD, Kundu D. An extension of generalized exponential distribution. Statistical Methodology 2011; 8: 485-496. https://doi.org/10.1016/j.stamet.2011.06.003

[13] Marshall AW, Olkin I. A new method for adding a parameter to a family of distributions with application to the exponential and Weibull families. Biometrika 1997; 84: 641-652. https://doi.org/10.1093/biomet/84.3.641

[14] Ghitany ME. Marshall-Olkin extended Pareto distribution and its application. International Journal of Applied Mathematics 2005; 18: 17-32. https://doi.org/10.1080/02664760500165008

[15] Ghitany ME, Al-Awadhi FA, Alkhalfan LA. Marshall-Olkin extended Lomax distribution and its application to censored data. Communication in Statistics - Theory and Methods 2007; 36: 1855-1866. https://doi.org/10.1080/03610920601126571

[16] Ghitany ME, Al-Hussaini EK, AlJarallah RA. Marshall-Olkin extended Weibull distribution and its application to censored data. Journal of Applied Statistics 2005; 32: 1025-1034.

[17] https://doi.org/10.1080/02664760500165008

[18] Ghitany ME, Kotz S. Reliability properties of extended linear failure-rate distributions. Probability in the Engineering and Informational Sciences 2007; 21: 441-450. https://doi.org/10.1017/S0269964807000071

[19] Zhang T, Xie M. Failure data analysis with extended Weibull distribution. Communication in Statistics - Simulation and Computation 2007; 36: 579-592. https://doi.org/10.1080/03610910701236081

[20] Caroni C. Testing for the Marshall-Olkin extended form of the Weibull distribution. Statistical Papers 2010; 51: 325-336. https://doi.org/10.1007/s00362-008-0172-x

[21] Mansoor M, Tahir MH, Cordeir GM, Serge P, Alzaatreh A. The Marshall-Olkin logistic-exponential distribution. Communications in Statistics-Theory and Methods, 2019; 48: 220-234. https://doi.org/10.1080/03610926.2017.1414254

[22] Ampadu CB. The Tan-G family of Distributions with illustration to Data in the health sciences, Physical Science and Biophysics journal 2019; 31: 1-3. https://doi.org/10.23880/PSBJ-16000125

[23] Ahsanullah M, Shakil M, Kibria BMG. On a Generalized Raised Cosine Distribution: Some Properties, Characterizations and Applications. Moroccan Journal of Pure and applied analysis 2019; 5: 63-85. https://doi.org/10.2478/mjpaa-2019-0006

[24] Eugene N, Lee C, Famoye F. Beta-normal distribution and its applications. Communications in Statistics-Theory and Methods 2002; 31: 497-512. https://doi.org/10.1081/STA-120003130

[25] Jones MC. Families of distributions arising from the distributions of order statistics. Test 2004; 13: 1-43. https://doi.org/10.1007/BF02602999

[26] Cordeiro GM, de Castro M. A new family of generalized distributions. Journal of Statistical Computation and Simulation 2011; 81: 883-893. https://doi.org/10.1080/00949650903530745

[27] Alexander C, Cordeiro GM, Ortega EMM, Sarabia JM. Generalized beta-generated distributions. Computational Statistics and Data Analysis 2012; 56: 1880-1897. https://doi.org/10.1016/j.csda.2011.11.015

[28] Morais AL, Barreto-Souza W. A compound class of Weibull and power series distributions. Computational Statistics and Data Analysis 2011; 55: 1410-1425. https://doi.org/10.1016/j.csda.2010.09.030

[29] Nadarajah S, Kotz S. The beta exponential distribution. Reliability Engineering and System Safety 2006; 91: 689-697. https://doi.org/10.1016/j.ress.2005.05.008

[30] Murthy DNP, Xie M, Jiang R. Weibull Models. Wiley, New York, 2004.

[31] Chen G, Balakrishnan N. A general purpose approximate goodness-of-fit test. Journal of Quality Technology 1995; 27: 154-161 https://doi.org/10.1080/00224065.1995.11979578

[32] Amini M, MirMostafaee SMTK, Ahmadi J. Log-gamma-generated families of distributions, Statistics 2014; 48: 913-932. https://doi.org/10.1080/02331888.2012.748775

[33] Ristic MM, Balakrishnan N. The gamma-exponentiated exponential distribution. Journal of Statistical Computation and Simulation 2012; 82: 1191-1206. https://doi.org/10.1080/00949655.2011.574633

[34] Torabi H, Montazari NH. The gamma-uniform distribution and its application, Kybernetika 2012; 48: 16-30.

[35] Torabi H, Montazari NH. The logistic-uniform distribution and its application, Communications in Statistics--Simulation and Computation 2014; 43: 2551-2569. https://doi.org/10.1080/03610918.2012.737491

[36] Nascimento ADC, Bourguignon M, Zea LM, Santos-Neto M, Silva RB, Cordeiro GM. The gamma extended-Weibull family of distributions. Journal of Statistical Theory and Applications 2014; 13: 1-16. https://doi.org/10.2991/jsta.2014.13.1.1

[37] Cordeiro GM, Ortega EMM, da Cunha DCC. The exponentiated generalized class of distributions. Journal of Data Science 2013; 11: 1-27. https://doi.org/10.6339/JDS.201301_11(1).0001

[38] Alzaatreh A, Lee C, Famoye F. A new method for generating families of continuous distributions. Metron 2013; 71: 63-79. https://doi.org/10.1007/s40300-013-0007-y

[39] Alzaghal A, Lee C, Famoye F. Exponentiated T-X family of distributions with some applications. International Journal of Probability and Statistics 2013; 2: 31-49. https://doi.org/10.5539/ijsp.v2n3p31

[40] Bourguignon M, Silva RB, Cordeiro GM. The Weibull-G family of probability distributions. Journal of Data Science 2014; 12: 53-68. https://doi.org/10.6339/JDS.201401_12(1).0004

[41] Cordeiro GM, Alizadeh M, Ortega EMM. The exponentiated half-logistic family of distributions: Properties and applications. Journal of Probability and Statistics Article ID 864396: 2014; 21 pages. https://doi.org/10.1155/2014/864396

[42] Aljarrah MA, Lee C, Famoye F. On generating T-X family of distributions using quantile functions. Journal of Statistical Distributions and Applications 2014; 1: Article 2. https://doi.org/10.1186/2195-5832-1-2

[43] Alzaatreh, A Ghosh I, Said H. On the gamma-logistic distribution. Journal of Modern Applied Statistical Methods 2014; 13: 55-70. https://doi.org/10.22237/jmasm/1398917040

[44] Tahir MH, Zubair M, Cordeiro GM, Alzaatreh A, Mansoor M. The Poisson-X family of distributions. Journal of Statistical Computation and Simulation 2016; 86: 2901-2921. https://doi.org/10.1080/00949655.2016.1138224

[45] Mansoor M, Tahir MH, Cordeir GM, Alzaatreh A, Zubair M. A new family of distributions to analyze lifetime data. Journal of Statistical Theory and Applications 2017; 16: 490--507. https://doi.org/10.2991/jsta.2017.16.4.6

[46] Mansoor M, Tahir MH, Cordeir GM, Alzaatreh A, Zubair M. The Poisson Weibull-X family of distributions. ProbStat Forum 2018; 11: 19-35.

[47] Mansoor M, Tahir MH, Cordeir GM, Ali S, Alzaatreh A. The Lindley negative-binomial distribution: properties, estimation and applications to lifetime data. Mathematica Slovaca 2020; 70. 917-934. https://doi.org/10.1515/ms-2017-0404

[48] Gupta RC, Gupta PI, Gupta RD. Modeling failure time data by Lehmann alternatives. Communications in Statistics-Theory and Methods 1998; 27: 887-904. https://doi.org/10.1080/03610929808832134

[49] Gupta RD, Kundu D. Generalized exponential distribution. Australian & New Zealand Journal of Statistics 1999; 41: 173-188. https://doi.org/10.1111/1467-842X.00072

[50] Gradshteyn I. S, Ryzhik IM. Tables of Integrals, Series, and Products. Academic Press, NewYork. 2000.

[51] R Development Core Team. R: A Language and Environment for Statistical Computing, R Foundation for Statistical Computing (Vienna, Austria), 2009.

[52] Nadarajah S. A truncated inverted beta distribution with application to air pollution data. Stochastic Environmental Research and Risk Assessment 2008; 2: 285-289. https://doi.org/10.1007/s00477-007-0120-7

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