# Homoclinic Solutions for Some Nonperiodic Fourth Order Differential Equations with Sublinear Nonlinearities

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**Abstract:** In this paper we investigate the existence of homoclinic solutions for the following fourth order nonautonomous differential equations;

$u^{(4)} + wu'' + a(x)u = f(x,u), (FDE)$

where $W$ is a constant, $a \in C(R,R)$ and $f \in C(R \times R, R)$. The novelty of this paper is that, when (FDE) is nonperiodic, i.e., $a$ and $f$ are nonperiodic in $x$, assuming that $a$ is bounded from below and $f$ is sublinear as $|u| \to +\infty$, we establish one new criterion to guarantee the existence and multiplicity of homoclinic solutions of (FDE). Recent results in the literature are generalized and improved.

**Keywords:** Homoclinic solutions, Critical point, Variational methods, Genus.

## 1. INTRODUCTION

In the present paper we deal with the existence of homoclinic solutions for the following nonperiodic fourth order nonautonomous differential equations;

$$u^{(4)} + wu'' + a(x)u = f(x,u), (FDE),$$

where $w$ is a constant, $a \in C(R,R)$ and $f \in C(R \times R, R)$. In (FDE), let $f(x,u)$ be of the form;

$$f(x,u) = b(x)u^2 + c(x)u^3,$$

then (FDE) reduces to the following equation;

$$u^{(4)} + wu'' + a(x)u - b(x)u^2 - c(x)u^3 = 0, \qquad (1.1)$$

which has been put forward as mathematical model for the study of pattern formation in physics and mechanics, see for instance [2, 4, 5, 7, 8, 9, 19] and the references there in.

For the problem of finding a homoclinic solution (i.e., a nontrivial solution $u(x)$ such that $u(x) \to 0$ as $|x| \to +\infty$) of the fourth order differential equations, we refer the reader to [1, 3, 14] concerned with the autonomous case. Compared to the autonomous case, the nonautonomous case seems to be more difficult, because of the lack of the translation invariance and the existence of a first integral. Tersian and Chaparova [19] showed that Eq.(1.1) possesses one nontrivial homoclinic solution by using the Mountain Pass Theorem when $a(x)$, $b(x)$ and $c(x)$ are continuous periodic functions and satisfy some other assumptions. Li [12] extended the results to some general nonlinear term, i.e., (FDE), assuming that $a(x)$ and $f(x,u)$ are periodic in $x$, and $f(x,u)$ satisfies the Ambrosetti-Rabinowitz condition ((AR) condition). Moreover, Li [13] dealt with the nonperiodic case of Eq.(1.1) and obtained the existence of nontrivial homoclinic solutions *via* using a compactness lemma and a mountain pass theorem. Sun and Wu [17] considered the following nonperiodic fourth order differential equations with a perturbation;

$$u^{(4)} + wu'' + a(x)u = f(x,u) + \lambda h(x)|u|^{p-2}u, \quad x \in R,$$

where $w$ is a constant, $\lambda > 0$ is a parameter, $a \in C(R,R)$, $f \in C(R \times R, R)$, $1 \le p < 2$ and $h \in L^{\frac{2}{2-p}}(R)$, which has been improved for some more generalized perturbation in [22]. More recently, Li *et al*. [10] studied the existence of infinitely many homoclinic solutions for nonperiodic (FDE) when the nonlinear term $f(x,u)$ satisfies the superlinear condition, but does not fulfil the well-known (AR) condition, see its Theorem 1.1. However we must point out that, for the case that (FDE) is nonperiodic, to obtain the existence of homoclinic solutions, the following coercive condition on $a$ is often needed:

• $a: R \to R$ is a continuous function, and there exists some constant $a_1 > 0$ such that;

$$0 < a_1 \le a(x) \to +\infty \quad as \quad |x| \to +\infty, \qquad (1.2)$$

which is used to establish the corresponding compact embedding lemmas on suitable functional spaces, see Lemma 2 in [13], Lemma 2.2 in [17] and Lemma 2.3 in [10].

It is obvious that, if $a$ is bounded, then it is not covered by (a). Inspired by the above facts, more

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recently, the authors [18, 21, 23] investigated the existence of homoclinic solutions of (FDE) for the case that $a$ is nonperiodic and bounded from below. Explicitly, assuming that the following conditions hold:

- $a \in C(R,R)$ is continuous and there exists a positive $\tau > 0$ such that;

$$a(x) \geq \tau > 0 \quad and \quad w \leq 2\sqrt{\tau};$$

- there exist a constant $1 < \vartheta < 2$ and positive function $b \in L^{\frac{2}{2-\vartheta}}(R,R^{+})$ such that;

$$|f(x,u)| \leq \vartheta b(x)|u|^{\vartheta-1}, \quad \forall (x,u) \in R \times R;$$

- there exist $x_0 \in R$ and $v \in (1,2)$ such that;

$$\liminf_{(x,u)\to(x_0,0)} \frac{F(x,u)}{|u|^{v}} > 0,$$

where $F(x,u) = \int_0^u f(x,t)dt$, then Yang [21] showed that (FDE) possesses at least one nontrivial homoclinic solution. If, in addition, $f$ is odd in $u$ variable, i.e.,

- $f(x,u) = -f(x,-u), \forall (x,u) \in R \times R$,

then (FDE) possesses infinitely many homoclinic solutions. In [18], Sun *et al*. considered the following nonperiodic fourth order differential equations with a parameter:

$$u^{(4)} + wu'' + \lambda a(x)u = f(x,u), \tag{1.3}$$

where $w$ is a constant, $\lambda > 0$ is a constant and $f \in C(R \times R, R)$. Assuming that the function $a$ satisfies the following conditions:

- $a \in C(R,R)$ and $a \geq 0$ on $R$; there exists $c > 0$ such that the set $\{a < c\} = \{x \in R | a(x) < c\}$ is nonempty and $|\{a < c\}| < c_0 S_{\infty}^{-2}$, where $|\cdot|$ is the Lebesgue measure, $S_{\infty}$ is the best Sobolev constant for the embedding of $H^2(R)$ in $L^{\infty}(R)$ and $c_0$ is given in Lemma 2.1;

- $T = int\, a^{-1}(0)$ is nonempty and $\overline{T} = a^{-1}(0)$ such that $T$ is a finite interval;

and $f$ is supposed to satisfy.

- there exist a constant $\vartheta \in (1,2)$ and a positive function $b \in L^{\xi}(R,R^{+})$ with $\xi \in (1, \frac{2}{2-\vartheta}]$ such that;

$$|f(x,u)| \leq b(x)|u|^{\vartheta-1} \quad for\ all\ (x,u) \in R \times R;$$

- there exist two constants $\eta, \delta > 0$ such that;

$$|F(x,u)| \geq \eta |u|^{\vartheta} \quad for\ all\ x \in T\ and\ u \in R\ with\ |u| \leq \delta,$$

then they showed that there exists $\Lambda_0 > 0$ such that for every $\lambda > \Lambda_0$, Eq.(1.3) has at least one homoclinic solution $u_{\lambda}$, and explored the phenomenon of concentration of homoclinic solutions as $\lambda \to \infty$, which has been generalized in recent paper [11] when the nonlinear term $f(x,u)$ satisfies the asymptotically linear condition, and the non-existence of nontrivial homoclinic solutions is also discussed. In [23], for the case that $a$ is bounded in the following sense;

- $a \in C(R,R)$ and there exits two constants $0 < \tau_1 < \tau_2 < +\infty$ such that;

$$0 < \tau_1 \leq a(t) \leq \tau_2 \quad for\ all\ x \in R,$$

and assuming that $f$ satisfies some superlinear condition weaker than (AR) condition, we showed that (FDE) has at least one nontrivial homoclinic solution.

Motivated by the above results, in present paper we focus our attention on the existence and multiplicity of homoclinic solutions of (FDE) for the case that $a$ is bounded from below and $f$ is sublinear as $|u| \to \infty$. Explicitly, we suppose that (A) is satisfied and $f$ fulfils;

- $f: R \times R \to R$ is continuous, there exist a constant $\vartheta \in (1,2)$ and a positive function $b \in L^{\xi}(R,R^{+})$ with $1 \leq \xi \leq 2/(2-\vartheta)$ such that;

$$|f(x,u)| \leq \vartheta b(x)|u|^{\vartheta-1} \quad for\ all\ (x,u) \in R \times R;$$

and ($F_2$).

Now we formulate our main result.

**Theorem 1.1** Under the assumptions of (A), $(F_1)'$ and $(F_2)$, (FDE) has at least one nontrivial homoclinic solution. In addition, if $f$ is odd in $u$ variable, that is, $(F_3)$ holds, then (FDE) possesses infinitely many nontrivial homoclinic solutions.

**Remark 1.2** In view of $F(x,0) = 0$ and $(F_1)'$, we have;

$$|F(x,u)| \leq b(x)|u|^{\vartheta}, \quad \forall (x,u) \in R \times R. \tag{1.4}$$

In $(F_1)'$, we assume that $b \in L^{\xi}(R,R^{+})$ with $1 \leq \xi \leq 2/(2-\vartheta)$, which is a generalization of $(F_1)$. Therefore, the results in [21] are extended. In [18], under the assumptions of $(V_1)$, $(V_2)$, $(D_1)$ and $(D_2)$, the authors investigated the existence of homoclinic solutions and the concentration of homoclinic solutions. In our Theorem 1.1, we deal with the existence and multiplicity of homoclinic solutions of (FDE). So the results in [18] are generalized.

**Remark 1.3** Recently, Chapiro *et al*. [6] considered the existence of homoclinic solutions for following nonperiodic semilinear fourth order differential equations;

$$u^{(4)} + wu'' + a(x)u = f(x,u,u'). \tag{1.5}$$

Under some reasonable hypotheses on $w$, $a$ and $f$, using the iterative method, they obtained the existence of at least one nontrivial homoclinic solution of Eq.(1.5). Although, the authors dealt with the case when the nonlinear term $f$ depends on $u$ and its derivative (which is a generalized nonlinearity), the assumption ($f_3$) (see its Theorem 2.1) implies that $f$ is superlinear with respect $u$. Here, in our Theorem 1.1 we consider (FDE) for the case that $f$ is sublinear as $|u| \to +\infty$ and obtain the existence of infinitely many homoclinic solutions.

The remaining part of this paper is structured as follows. Some preliminary results are presented in Section 2. In Section 3, we are devoted to accomplishing the proof of Theorem 1.1.

## 2. PRELIMINARY RESULTS

In order to prove Theorem 1.1 *via* the critical point theory, we firstly describe some properties of the space $E$ on which the variational framework associated with (FDE) is defined.

**Lemma 2.1** ([19, Lemma 8]) Assume that $a(x) \geq \tau > 0$ and $w \leq 2\sqrt{\tau}$. Then there exists a constant $c_0 > 0$ such that;

$$\int_R [u''(x)^2 - wu'(x)^2 + a(x)u(x)^2]dx \geq c_0 PuP_{H^2}^2 \quad \textit{for all} \quad u \in H^2(R), \tag{2.1}$$

where $PuP_{H^2} = (\int_R [u''(x)^2 + u'(x)^2 + u(x)^2]dx)^{1/2}$ is the norm of Sobolev space $H^2(R) = W^{2,2}(R)$ as usual.

Due to Lemma 2.1, we define;

$$E = \left\{ u \in H^2(R) : \int_R [u''(x)^2 - wu'(x)^2 + a(x)u(x)^2]dx < +\infty \right\},$$

with the inner product;

$$(u,v) = \int_R [u''(x)v''(x) - wu'(x)v'(x) + a(x)u(x)v(x)]dx$$

and the corresponding norm;

$$PuP = (\int_R [u''(x)^2 - wu'(x)^2 + a(x)u(x)^2]dx)^{1/2}.$$

Then, it is easy to verify that $E$ is a Hilbert space. Note that;

$$H^2(R) \subset L^p(R), \quad 2 \leq p \leq +\infty$$

and the embedding is continuous. That is, there exists a constant $C_p > 0$ such that;

$$PuP_p \leq C_p PuP_{H^2}, \quad \forall u \in E, \tag{2.2}$$

for any $p \in [2, +\infty]$. Combining (2.2) with (2.1), for any $p \in [2, +\infty]$, there is an another constant (still denoted by $C_p$) such that;

$$PuP_p \leq C_p PuP, \quad \forall u \in E. \tag{2.3}$$

Here $L^p(R)$ ($2 \leq p < +\infty$) denotes the Banach spaces of functions on $R$ with values in $R$ under the norm;

$$PuP_p := (\int_R |u(x)|^p \, dx)^{1/p}.$$

$L^\infty(R)$ is the Banach space of essentially bounded functions from $R$ into $R$ equipped with the norm;

$$PuP_\infty := ess\sup\{|u(x)| : x \in R\}.$$

To deal with the existence of homoclinic solutions of (FDE), we appeal to the following well-known result, see for example [15].

**Definition 2.2** $I \in C^1(B,R)$ is said to satisfy the (PS) condition if any sequence $\{u_j\}_{j\in N} \subset B$, for which $\{I(u_j)\}_{j\in N}$ is bounded and $I'(u_j) \to 0$ as $j \to +\infty$, possesses a convergent subsequence in *B*.

**Lemma 2.3** Let B be a real Banach space and $I \in C^1(B,R)$ satisfying the (PS) condition. If I is bounded from below, then $c = \inf_B I(u)$ is a critical value of $I$.

To obtain the existence of infinitely many homoclinic solutions of (FDE) under the assumptions of Theorem 1.1, we shall employ the “genus" properties in critical point theory, see [15, 16].

Let B be Banach space, $I \in C^1(B,R)$ and $c \in R$. We set;

$$\Sigma \quad = \{A \subset B - \{0\} : A \text{ is closed in } B \text{ and symmetric with respect to } 0\},$$

$$K_c \quad = \{u \in B : I(u) = c, I'(u) = 0\}, \quad I^c = \{u \in B : I(u) \le c\}.$$

**Definition 2.4** For $A \in \Sigma$, we say the genus of $A$ is $j$ (denoted by $\gamma(A) = j$) if there is an odd map $\psi \in C(A, R^j \setminus \{0\})$ and $j$ is the smallest integer with this property.

**Lemma 2.5** [16, Theorem 2.1] Let $I$ be an even $C^1$ functional on B and satisfy the (PS) condition. For any $j \in N$, set;

$$\Sigma_j = \{A \in \Sigma : \gamma(A) \ge j\}, \quad c_j = \inf_{A \in \Sigma_j} \sup_{u \in A} I(u).$$

• If $\Sigma_j \ne \varnothing$ and $c_j \in R$, then $c_j$ is a critical value of $I$;

• if there exists $r \in N$ such that;

$$c_j = c_{j+1} = \cdots = c_{j+r} = c \in R,$$

and $c \ne I(0)$, then $\gamma(K_c) \ge r+1$.

**Remark 2.6** From Remark 7.3 in [15], we know that if $K_c \in \Sigma$ and $\gamma(K_c) > 1$, then $K_c$ contains infinitely many distinct points, i.e., $I$ has infinitely many distinct critical points in *B*.

## 3. PROOF OF THEOREM 1.1

Now we are going to establish the corresponding variational framework to obtain the existence and multiplicity of homoclinic solutions of (FDE). To this end, define the functional $I : B = E \to R$ by,

$$\begin{aligned} I(u) &= \int_R [\frac{1}{2} u''(x)^2 - \frac{1}{2} wu'(x)^2 + \frac{1}{2} a(x)u(x)^2 - F(x,u(x))]dx \\ &= \frac{1}{2} PuP^2 - \int_R F(x,u(x))dx. \end{aligned} \tag{3.1}$$

The purpose of this section is to prove Theorem 1.1. To this aim, we present some lemmas which will be used in the subsequent discussion.

**Lemma 3.1** Under the conditions of Theorem 1.1, $I \in C^1(E,R)$, i.e., $I$ is a continuously Fr $e'$ chet-differentiable functional defined on $E$. Moreover, we have;

$$\begin{aligned} I'(u)v = \int_R [u''(x)v''(x) - wu'(x)v'(x) + \\ a(x)u(x)v(x) - f(x,u(x))v(x)]dx \end{aligned} \tag{3.2}$$

for all $u$, $v \in E$, which yields that;

$$I'(u)u = PuP^2 - \int_R f(x,u(x))u(x)dx. \tag{3.3}$$

**Proof.** We firstly show that $I : E \to R$. By the Hölder inequality, (1.4) and (2.3), we have;

$$\begin{aligned} 0 \le \int_R | F(x,u(x)) | dx \le \int_R | b(x) || u(x) |^{\vartheta} d \\ x \le PbP_\xi PuP^{\vartheta}_{\vartheta\xi^*} \le C^{\vartheta}_{\vartheta\xi^*} PbP_\xi PuP^{\vartheta}, \end{aligned} \tag{3.4}$$

where $\xi^*$ is the conjugate exponent of $\xi$, i.e., $1 = \frac{1}{\xi} + \frac{1}{\xi^*}$, $C_{\vartheta\xi^*}$ is as defined in (2.3). Combining this with (3.1), we see that $I : E \to R$.

Next we prove that $I \in C^1(E,R)$. To this end, we rewrite $I = A - B$ as follows:

$$A(u) = \frac{1}{2} PuP^2, \quad B(u) = \int_R F(x,u(x))dx. \tag{3.5}$$

It is easy to check that $A \in C^1(E,R)$, and we have;

$$A'(u)v = \int_R [u''(x)v''(x) - wu'(x)v'(x) + a(x)u(x)v(x)]dx.$$

Therefore, it is sufficient to show that this is the case for $B$. In the process we shall see that $B \in C^1(E,R)$ and;

$$B'(u)v = \int_R f(x,u(x))v(x)dx, \tag{3.6}$$

which is defined for all $u$, $v \in E$.

For any given $u \in E$, let us define $J(u) : E \to R$ by,

$$J(u)v = \int_R f(x,u(x))v(x)dx, \quad \forall v \in E. \tag{3.7}$$

It is obvious that, for any given $u \in E$, $J(u)$ is a linear functional. Moreover, $J(u)$ is also bounded. Indeed, for any given $u \in E$, in view of (2.3), the Young inequality and the Hölder inequality, we obtain that;

$$\begin{aligned} PJ(u)P &= \sup_{PvP=1} | J(u)v | = \sup_{PvP=1} | \int_R f(x,u(x))v(x)dx | \\ &\le \sup_{PvP=1} \vartheta \int_R | b(x) || u(x) |^{\vartheta-1} | v(x) | dx \\ &\le \sup_{PvP=1} \vartheta \int_R | b(x) | (\frac{\vartheta-1}{\vartheta} | u(x) |^{\vartheta} + \frac{1}{\vartheta} | v(x) |^{\vartheta}) dx \\ &\le \sup_{PvP=1} [(\vartheta-1) PbP_\xi PuP^{\vartheta}_{\vartheta\xi^*} + PbP_\xi PvP^{\vartheta}_{\vartheta\xi^*}] \\ &\le \sup_{PvP=1} [(\vartheta-1) PbP_\xi C^{\vartheta}_{\vartheta\xi^*} PuP + PbP_\xi C^{\vartheta}_{\vartheta\xi^*} PvP] \\ &= (\vartheta-1) PbP_\xi C^{\vartheta}_{\vartheta\xi^*} PuP + PbP_\xi C^{\vartheta}_{\vartheta\xi^*}. \end{aligned} \tag{3.8}$$

In what follows, we show that $J$ is the Gateaux derivative of $B$ defined in (3.5). For $u, v \in E$, using the Mean Value Theorem, we have;

$$\int_R F(x,u(x)+tv(x))dx - \int_R F(x,u(x))dx = \int_R f(x,u(x)+th(x)v(x))tv(x)dx$$

for some $h(x) \in (0,1)$, where $0 < |t| < 1$. For any $\varepsilon > 0$, in view of $b \in L^{\xi}(R,R^+)$, there exists an $R > 0$ such that;

$$\left(\int_{|x|>R} |b(x)|^{\xi} dx\right)^{1/\xi} < \varepsilon. \tag{3.9}$$

For the given $R > 0$ in (3.9), based on $(F_1)'$ and (2.3), we have

$$\frac{1}{t}\left[\int_R f(x,u(x)+th(x)v(x))tv(x)dx - \int_R f(x,u(x))tv(x)dx\right]$$

$$= \int_R (f(x,u(x)+th(x)v(x)) - f(x,u(x)))v(x)dx$$

$$= \int_{|x|\le R} (f(x,u(x)+th(x)v(x)) - f(x,u(x)))v(x)dx$$

$$+ \int_{|x|>R} (f(x,u(x)+th(x)v(x)) - f(x,u(x)))v(x)dx$$

$$\le \left(\int_{|x|\le R} |f(x,u(x)+th(x)v(x)) - f(x,u(x))|^2 dx\right)^{1/2} \left(\int_{|x|\le R} |v(x)|^2 dx\right)^{1/2}$$

$$+ \vartheta \int_{|x|>R} |b(x)| (2|u(x)|^{\vartheta-1} + |v(x)|^{\vartheta-1}) |v(x)| dx$$

$$\le C_2 \|v\| \left(\int_{|x|\le R} |f(x,u(x)+th(x)v(x)) - f(x,u(x))|^2 dx\right)^{1/2}$$

$$+ \left(\int_{|x|>R} |b(x)|^{\xi} dt\right)^{1/\xi} \left[2(\vartheta-1)\left(\int_{|x|>R} |u(x)|^{\vartheta\xi^*} dx\right)^{1/\xi^*} + \left(\int_{|x|>R} |v(x)|^{\vartheta\xi^*} dx\right)^{1/\xi^*}\right]$$

$$+ \vartheta \left(\int_{|x|>R} |b(x)|^{\xi} dt\right)^{1/\xi} \left(\int_{|x|>R} |v(x)|^{\vartheta\xi^*} dx\right)^{1/\xi^*}$$

$$\le C_2 \|v\| \left(\int_{|x|\le R} |f(x,u(x)+th(x)v(x)) - f(x,u(x))|^2 dx\right)^{1/2}$$

$$+ \vartheta \left(\int_{|x|>R} |b(x)|^{\xi} dt\right)^{1/\xi} \left[2(\vartheta-1)\left(\int_R |u(x)|^{\vartheta\xi^*} dx\right)^{1/\xi^*} + 2\left(\int_R |v(x)|^{\vartheta\xi^*} dx\right)^{1/\xi^*}\right].$$

For the first term of the above inequality, due to the fact that $E|_{[-R,R]}$ is compactly embedded into $L^{\infty}([-R,R],R)$, it can be made arbitrary small by choosing $|t|$ small enough. As far as the second term of the above inequality is concerned, according to (2.3) and (3.9), we have;

$$\vartheta \left(\int_{|x|>R} |b(x)|^{\xi} dt\right)^{1/\xi} \left[2(\vartheta-1)\left(\int_R |u(x)|^{\vartheta\xi^*} dx\right)^{1/\xi^*} + 2\left(\int_R |v(x)|^{\vartheta\xi^*} dx\right)^{1/\xi^*}\right]$$

$$\le \vartheta \left(\int_{|x|>R} |b(x)|^{\xi} dt\right)^{1/\xi} \left[2(\vartheta-1) C^{\vartheta}_{\vartheta\xi^*} \|u\|^{\vartheta}_{\vartheta\xi^*} + 2C^{\vartheta}_{\vartheta\xi^*} \|v\|^{\vartheta}_{\vartheta\xi^*}\right]$$

$$\le \varepsilon\vartheta \left[2(\vartheta-1) C^{\vartheta}_{\vartheta\xi^*} \|u\|^{\vartheta}_{\vartheta\xi^*} + 2C^{\vartheta}_{\vartheta\xi^*} \|v\|^{\vartheta}_{\vartheta\xi^*}\right],$$

which implies that the second term can be also made arbitrary small. Therefore, we show that $J$ is the Gateaux derivative of $B$.

Next it is sufficient to prove that $B'$ is continuous (which implies that $J$ is the Fréchet derivative of $B$, see Proposition 1.3 in [20]). Suppose that $u \to u_0$ in $E$, then, in view of (2.3) and (3.9), it deduces that;

$$\sup_{\|v\|=1} |B'(u)v - B'(u_0)v|$$

$$= \sup_{\|v\|=1} \left|\int_R (f(x,u(x)) - f(x,u_0(x)))v(x)dx\right|$$

$$\le \sup_{\|v\|=1} \int_{|x|\le R} |(f(x,u(x)) - f(x,u_0(x)))v(x)| dx$$

$$+ \sup_{\|v\|=1} \int_{|x|>R} \vartheta |b(x)| (|u(x)|^{\vartheta-1} + |u_0(x)|^{\vartheta-1}) |v(x)| dx$$

$$\le \sup_{\|v\|=1} \left(\int_{|x|\le R} |f(x,u(x)) - f(x,u_0(x))|^2 dx\right)^{1/2} \left(\int_{|x|\le R} |v(x)|^2 dx\right)^{1/2}$$

$$+ \sup_{\|v\|=1} \left(\int_{|x|>R} |b(x)|^{\xi} dx\right)^{1/\xi} (\vartheta-1)\left[\left(\int_{|x|>R} |u(x)|^{\vartheta\xi^*} dx\right)^{1/\xi^*} + \left(\int_{|x|>R} |u_0(x)|^{\vartheta\xi^*} dx\right)^{1/\xi^*}\right]$$

$$+ \sup_{\|v\|=1} 2\left(\int_{|x|>R} |b(x)|^{\xi} dx\right)^{1/\xi} \left(\int_{|x|>R} |v(x)|^{\vartheta\xi^*} dx\right)^{1/\xi^*}$$

$$\le C_2 \left(\int_{|x|\le R} |f(x,u(x)) - f(x,u_0(x))|^2 dx\right)^{1/2}$$

$$+ C^{\vartheta}_{\vartheta\xi^*}\left[(\vartheta-1)(\|u\|^{\vartheta}_{\vartheta\xi^*} + \|u_0\|^{\vartheta}_{\vartheta\xi^*}) + 2\right] \left(\int_{|x|>R} |b(x)|^{\xi} dx\right)^{1/\xi}$$

$$\le \varepsilon C_2 + \varepsilon C^{\vartheta}_{\vartheta\xi^*}\left[(\vartheta-1)(\|u\|^{\vartheta}_{\vartheta\xi^*} + \|u_0\|^{\vartheta}_{\vartheta\xi^*}) + 2\right],$$

which yields that $B'(u) - B'(u_0) \to 0$ as $u \to u_0$. Thus, $B'$ is continuous. Therefore, we have shown that $I \in C^1(E,R)$.

**Lemma 3.2** If $(A)$ and $(F_1)'$ hold, then $I$ satisfies the (PS) condition.

**Proof.** Assume that $\left\{u_j\right\}_{j\in N} \subset E$ is a sequence such that $\left\{I(u_j)\right\}_{j\in N}$ is bounded and $I'(u_j) \to 0$ as $j \to +\infty$. Then there exists a constant $C>0$ such that;

$$|I(u_j)| \leq C, \tag{3.10}$$

for every $j \in N$. We firstly prove that $\left\{u_j\right\}_{j\in N}$ is bounded in $E$. From (3.1), (??) and (??), it is easy to deduce that;

$$\begin{aligned} Pu_jP^2 &= 2I(u_j) + 2\int_R F(x,u_j(x))dx \\ &\leq 2C + 2C^{\vartheta}_{\vartheta\xi^*} PbP_{\xi} Pu_jP^{\vartheta}. \end{aligned} \tag{3.11}$$

Since $1<\vartheta<2$, the inequality (??) shows that $\left\{u_j\right\}_{j\in N}$ is bounded in $E$. Then the sequence $\left\{u_j\right\}_{j\in N}$ has a subsequence, again denoted by $\left\{u_j\right\}_{j\in N}$, and there exists $u \in E$ such that;

$$u_j \dagger \; u \; weakly \; in \; E,$$

which implies that;

$$(I'(u_j) - I'(u))(u_j - u) \to 0 \quad as \; j \to +\infty, \tag{3.12}$$

and, for any $j \in N$, there exists a constant $M>0$ such that;

$$Pu_jP_{\infty} \leq C_{\infty} Pu_jP \leq M \quad and \quad PuP_{\infty} \leq C_{\infty} PuP \leq M. \tag{3.13}$$

Next, we verify that $I$ satisfies the (PS) condition. For the given $R>0$ in (3.9), on account of the continuity of $f(x,u)$ and $u_j \to u$ in $L^{\infty}_{loc}(R,R)$, there exists $j_0 \in$ N such that;

$$\int_{|x|\leq R} (f(x,u_j(x)) - f(x,u(x)))(u_j(x) - u(x))dx < \varepsilon \quad for \; j \geq j_0. \tag{3.14}$$

On the other hand, joining $(F_1)'$, (2.3), (3.9) and (3.13), we obtain that;

$$\begin{aligned} &\int_{|x|>R} (f(x,u_j(x)) - f(x,u(x)))(u_j(x) - u(x))dx \\ &\leq \int_{|x|>R} | f(x,u_j(x)) - f(x,u(x)) || u_j(x) - u(x) | dx \\ &\leq \vartheta \int_{|x|>R} | b(x) | (| u_j(x) |^{\vartheta-1} + | u(x) |^{\vartheta-1})(| u_j(x) | + | u(x) |)dx \\ &\leq 2\vartheta \int_{|x|>R} | b(x) | (| u_j(x) |^{\vartheta} + | u(x) |^{\vartheta})dx \\ &\leq 2\vartheta (\int_{|x|>R} | b(x) |^{\xi} dx)^{1/\xi} (Pu_jP^{\vartheta}_{\vartheta\xi^*} + PuP^{\vartheta}_{\vartheta\xi^*}) \\ &\leq 2\vartheta (\int_{|x|>R} | b(x) |^{\xi} dx)^{1/\xi} C^{\vartheta}_{\vartheta\xi^*} (Pu_jP^{\vartheta} + PuP^{\vartheta}) \\ &\leq 4\varepsilon\vartheta C^{\vartheta}_{\vartheta\xi^*} (\frac{M}{C_{\infty}})^{\vartheta}. \end{aligned} \tag{3.15}$$

Since $\varepsilon > 0$ is arbitrary, combining (3.14) with (3.15), we get;

$$\int_R (f(x,u_j(x)) - f(x,u(x)))(u_j(x) - u(x))dx \to 0 \tag{3.16}$$

as $j \to +\infty$. Consequently, in view of (??), (3.16) and the following equality;

$$(I'(u_j) - I'(u), u_j - u) = \begin{array}{l} Pu_j - uP^2 \\ -\int_R (f(x,u_j(x)) - f(x,u(x)))(u_j(x) - u(x))dx, \end{array}$$

it is obvious that $Pu_j - uP \to 0$ as $j \to +\infty$. That is, the proof is completed.

Now we can formulate the proof of Theorem 1.1.

**Proof of Theorem 1.1** It is clear that $I(0)=0$, and by Lemma 3.2 we have known that $I$ is a $C^1$ functional on $E$ satisfying the (PS) condition. On the other hand, in view of (3.1) and (??), we obtain that;

$$I(u) \geq \frac{1}{2} PuP^2 - C^{\vartheta}_{\vartheta\xi^*} PbP_{\xi} PuP^{\vartheta}, \tag{3.17}$$

which implies that $I$ is bounded below on $E$. Hence by Lemma 2.3, $c = \inf_E I(u)$ is a critical value of $I$, namely, there is a critical point $u^* \in E$ such that $I(u^*) = c$ and $I'(u^*) = 0$. Moreover, this critical value $c$ is a negative real number as the following argument will show, and so $u^*$ is a nontrivial homoclinic solution.

In what follows, we investigate the existence of infinitely many homoclinic solutions of (FDE). From $(F_3)$, it is obvious that $I$ is even. In order to apply Lemma 2.5, we prove that

$$for \; any \; j \in N \; there \; exists \; \varepsilon > 0 \; such \; that \; \gamma(I^{-\varepsilon}) \geq j. \tag{3.18}$$

By $(F_2)$, there exist an open set $D \subset R$ with $x_0 \in D$, $\sigma > 0$ and $\eta > 0$ such that;

$$F(x,u) \geq \eta |u|^{\nu}, \quad \forall (x,u) \in D \times R, |u| \leq \sigma. \tag{3.19}$$

For any $j \in N$, we take $j$ disjoint open sets $D_i$ such that $\bigcup_{i=1}^{j} D_i \subset D$. For $i=1,2,\ldots,j$, let $u_i \in (W_0^{2,2}(D_i) \cap E) \setminus \{0\}$ with $Pu_iP = 1$, and;

$$E_j = span\{u_1, u_2, \ldots, u_j\}, \quad S_j = \{u \in E_j : PuP = 1\}.$$

Then, for any $u \in E_j$, there exist $\lambda_i \in R$, $i=1,2,\ldots,j$ such that;

$$u(x)=\sum_{i=1}^{j}\lambda_i u_i(x) \quad for\ x\in R. \tag{3.20}$$

From (??), it follows that;

$$\|u\|_v=(\int_R |u(x)|^v)^{1/v}=(\sum_{i=1}^{j}|\lambda_i|^v\int_{D_i}|u_i(x)|^v\,dx)^{1/v} \tag{3.21}$$

and,

$$\begin{aligned}\|u\|^2 &= \int_R[u''(x)^2-wu'(x)^2+a(x)u(x)^2]dx\\ &=\sum_{i=1}^{j}\lambda_i^2\int_{D_i}[u_{i''}(x)^2-wu_{i'}(x)^2+a(x)u_i(x)^2]dx\\ &=\sum_{i=1}^{j}\lambda_i^2\int_{R}[u_{i''}(x)^2-wu_{i'}(x)^2+a(x)u_i(x)^2]dx\\ &=\sum_{i=1}^{j}\lambda_i^2\|u_i\|^2=\sum_{i=1}^{j}\lambda_i^2.\end{aligned} \tag{3.22}$$

Since all norms of a finite dimensional norm space are equivalent, there is a constant $\rho=\rho(j)>0$ such that;

$$\rho\|u\|\le\|u\|_v,\quad \forall u\in E_j. \tag{3.23}$$

Note that $F(t,0)=0$, and so according to (??)-(??), we have;

$$\begin{aligned}I(su) &= \frac{s^2}{2}\|u\|^2-\int_R F(x,su(x))dx\\ &=\frac{s^2}{2}\|u\|^2-\sum_{i=1}^{j}\int_{D_i}F(x,s\lambda_i u_i(x))dx\\ &\le\frac{s^2}{2}\|u\|^2-\eta s^v\sum_{i=1}^{j}|\lambda_i|^v\int_{D_i}|u_i(x)|^v\,dx\\ &=\frac{s^2}{2}\|u\|^2-\eta s^v\|u\|_v^v\\ &\le\frac{s^2}{2}\|u\|^2-\eta(\rho s)^v\|u\|^v\\ &=\frac{s^2}{2}-\eta(\rho s)^v\end{aligned} \tag{3.24}$$

for all $u\in S_j$ and sufficient small $s>0$. In this case (??) is applicable, since $u$ is continuous on $\overline{D}$ and so $|s\lambda_i u_i(x)|\le\sigma,\forall x\in D, i=1,2,\cdots,j$ can be true for sufficiently small $s$. Therefore, it follows from (??) that there exist $\varepsilon>0$ and $\delta>0$ such that;

$$I(\delta u)<-\varepsilon \quad for\ u\in S_j. \tag{3.25}$$

Let;

$$S_j^\delta=\{\delta u: u\in S_j\},$$

$$\Omega=\{(\lambda_1,\lambda_2,\ldots,\lambda_j)\in R^j:\sum_{i=1}^{j}\lambda_i^2<\delta^2\}.$$

Then it follows from (??) that;

$$I(u)<-\varepsilon,\quad \forall u\in S_j^\delta,$$

which, together with the fact that $I$ is an even $C^1$ functional on $E$, yields that;

$$S_j^\delta\subset I^{-\varepsilon}\in\Sigma,$$

where $I^{-\varepsilon}$ and $\Sigma$ have been previously introduced in Section 2. On the other hand, it follows from (??) and (??) that there exists an odd homeomorphism $\psi\in C(S_j^\delta,\partial\Omega)$. By some properties of the genus (see $3^\circ$ of Propositions 7.5 and 7.7 in [15]), we infer that;

$$\gamma(I^{-\varepsilon})\ge\gamma(S_j^\delta)=j, \tag{3.26}$$

so (??) follows. Set;

$$c_j=\inf_{A\in\Sigma_j}\sup_{u\in A}I(u),$$

where $\Sigma_j$ is defined in Lemma 2.5. It follows from (??) and the fact that $I$ is bounded from below on $E$ (see (3.17)), we have $-\infty<c_j\le-\varepsilon<0$, which implies that, for any $j\in N$, $c_j$ is a real negative number. By lemma 2.5 and Remark 2.6, $I$ has infinitely many nontrivial critical points. Consequently, (FDE) possesses infinitely many homoclinic solutions.

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