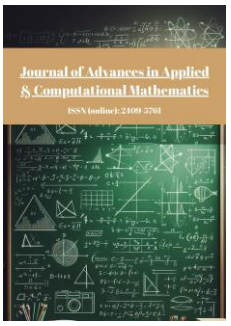
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Fractional Chebyshev-Type Inequalities for Synchronous Functions via Unified Fractional Remainder Operator

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ABSTRACT

In this paper, we establish new fractional Chebyshev-type inequalities for synchronous functions by means of a unified fractional remainder operator. Motivated by the classical Chebyshev integral inequality and its extensions, we investigate how such inequalities can be formulated within a generalized fractional integral framework. To this end, we introduce a normalized form of the unified fractional remainder operator and define an associated fractional Chebyshev functional. Using the nonnegativity and normalization properties of the corresponding fractional kernel, we derive a double-integral representation of this functional. This representation provides the key tool for establishing a fractional analogue of the classical two-function Chebyshev integral inequality for synchronous functions. Furthermore, we extend the approach to obtain a multi-function Chebyshev-type inequality within the same fractional framework. The assumptions ensuring the integrability of the functions and their products are taken into account in establishing these results. Finally, we demonstrate that the classical Chebyshev integral inequality is recovered as a particular case through an appropriate choice of the parameters of the unified operator. These findings extend the applicability of Chebyshev-type inequalities to the proposed fractional setting and provide a framework for deriving related inequalities.

Mathematics Subject Classification: 26A33; 26D07; 26D10; 33B15.

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1. Introduction

Fractional calculus is a natural extension of classical calculus which permits arbitrary real and more generally complex values for the orders of differentiation and integration. The current theory garnered significant attention due to its ability to explain nonlocal and memory-dependent phenomena, even though the topic originated in the early development of differential and integral calculus. Fractional operators are especially useful for modeling processes with hereditary features because, in contrast to standard integer-order operators, they incorporate information from an interval rather than from single point. Thus, finding applications in a variety of fields, including control theory, physics, engineering, mathematics, viscoelasticity, anomalous diffusion, and other applied sciences. For instance, fractional differential equations have been investigated using the homotopy analysis method for both nonlinear and linear partial differential equations, as well as other analytical approaches for fractional-order equations [1-3]. Related studies have explored exact solutions of nonlinear fractional evolution equations and conformable fractional models [4, 5], while further applications include fractional soliton models [6], epidemic models formulated using fractional derivatives [7-9], and fractional models of cancer chemotherapy [10]. More recently, variable-order fractional derivatives have been considered in the study of fractional logistic equations on finite intervals [11]. Related work has also examined generalized refinements of the Hermite–Hadamard inequality, further broadening the study of inequalities associated with generalized integral operators [12]. These developments illustrate the continuing interest in fractional operators across a range of theoretical and applied problems and motivate further investigation of their analytical properties, including their role in establishing integral inequalities.

Fractional integrals are among the most critical objects in fractional calculus, and several definitions have been introduced such as the Riemann–Liouville, Hadamard, Erdélyi–Kober, and other generalized fractional integrals. For example, the left and right hand Riemann-Liouville fractional integrals of order $\alpha > 0$ are given by [13]

$$\mathrm{RL}_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\,dt, \tag{1.1}$$

and

$$\mathrm{RL}_{b^-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)\,dt, \tag{1.2}$$

respectively. These operators have had a significant impact on the study of mathematical inequalities, offering a suitable foundation for extending classical integral estimates to nonlocal environments. Fractional kernels, in particular, enhance flexibility by enabling function values spanning an interval to be weighted based on their fractional ordering. This method not only recovers various classical inequalities as special cases, but also generates new limits that represent the nonlocal nature of fractional integration. These findings show that fractional integral operators are more than just extensions of classical integrals, they also give excellent analytical tools for producing improved and generalized versions of well-known inequalities.

Among the classical integral inequalities, Chebyshev's integral inequality occupies an important place in the theory of inequalities involving integral means. It provides a fundamental relationship between the integral of the product of two functions and the product of their respective integral means. More precisely, if $f, g: [a,b] \to R$ integrable, and synchronous functions, that is,

$$(f(x) - f(y))(g(x) - g(y)) \geq 0, \qquad x, y \in [a,b], \tag{1.3}$$

then Chebyshev's integral inequality states that

$$\frac{1}{b-a} \int_a^b f(x)\, g(x)\,dx \geq \frac{1}{(b-a)^2} \left(\int_a^b f(x)\,dx \right) \left(\int_a^b g(x)\,dx \right). \tag{1.4}$$

Equivalently, if the Chebyshev functional is defined by [14]

$$\mathrm{T}(f,g)=\frac{1}{b-a}\int_a^b f(x)g(x)dx-\frac{1}{(b-a)^2}\left(\int_a^b f(x)dx\right)\left(\int_a^b g(x)dx\right), \tag{1.5}$$

then synchrony of f and g implies

$$\mathrm{T}(f,g)\geq 0.$$

A useful representation of this functional is

$$\mathrm{T}(f,g)=\frac{1}{2(b-a)^2}\int_a^b\int_a^b(f(x)-f(y))(g(x)-g(y))dxdy. \tag{1.6}$$

More generally, for any finite collection of nonnegative and mutually synchronous functions, the Chebyshev functional admits the following multifunctional form

$$\mathrm{T}(f_1,\dots,f_m)=\frac{1}{b-a}\int_a^b\prod_{i=1}^m f_i(t)dt-\prod_{i=1}^m\left(\frac{1}{b-a}\int_a^b f_i(t)dt\right), \tag{1.7}$$

similarly, giving rise to the following classical multifunctional Chebyshev inequality

$$\frac{1}{b-a}\int_a^b\prod_{i=1}^m f_i(t)dt\geq\prod_{i=1}^m\left(\frac{1}{b-a}\int_a^b f_i(t)dt\right). \tag{1.8}$$

In [15], Belarbi and Dahmani obtained the following result about Chebyshev's inequality using Riemann-Liouville fractional integral operators

$$\mathrm{RL}^{\alpha}(fg)(x)\geq\frac{\Gamma(\alpha+1)}{x^{\alpha}}\mathrm{RL}^{\alpha}f(x)\mathrm{RL}^{\alpha}g(x). \tag{1.9}$$

Several authors have formulated and validated multiple novel integral inequalities of type (1.4) for the Chebyshev functional (1.5) by employing diverse fractional integral operators, as referenced in [16-24].

Recently in [25], Bouharket *et al*. used the k-weighted fractional integral of functions with respect to another function defined by

$${}_{a^+}I_{k,\omega}^{\alpha,\mu}f(x)=\frac{1}{\omega(x)\Gamma_k(x)}\int_a^x\mu'(s)(\mu(x)-\mu(s))^{\frac{\alpha}{k}-1}\omega(s)f(s)ds,\quad x>a,$$

and obtained a generalized Chebyshev-type fractional integral inequality

$$\frac{\omega(b)(\mu(x)-\mu(a))^{\frac{\beta}{k}}}{\omega(a)\Gamma_k(\alpha+k)}\ {}_{a^+}I_{k,\omega}^{\alpha,\mu}(fg)(x)+\frac{\omega(b)(\mu(x)-\mu(a))^{\frac{\alpha}{k}}}{\omega(a)\Gamma_k(\alpha+k)}\ {}_{a^+}I_{k,\omega}^{\beta,\mu}(fg)(x)$$

$$\geq\ {}_{a^+}I_{k,\omega}^{\alpha,\mu}(f)(x)\ {}_{a^+}I_{k,\omega}^{\beta,\mu}(g)(x)\ +\ {}_{a^+}I_{k,\omega}^{\beta,\mu}(f)(x)\ {}_{a^+}I_{k,\omega}^{\alpha,\mu}(g)(x),\quad x>a. \tag{1.10}$$

Despite the significant development of fractional integral inequalities, many current results are stated within certain fractional integral settings and hence are directly related to a specific choice of kernel, order, and endpoint (for example, $t-s$ or $s-t$). A particularly interesting direction in this field is the development of a unified formulation that scales with the overall interval length $b-a$ and can include various previously known integral expressions as specific cases by making appropriate parameter choices. Thus, further investigation is needed to establish such inequalities within this specific framework. In particular, the role of the associated normalized kernel in deriving two-function and multifunction Chebyshev-type inequalities deserves further attention.

To address the research gap, we investigate unified fractional operators whose kernels have a Beta-type structure. The dependence on the interval length and the Gamma-function normalization allow these operators to be interpreted as generalized fractional weighted integrals. After normalization, the associated kernel is nonnegative and has unit integral, making it suitable for studying integral inequalities involving products of

functions and their weighted means. Thus, we establish new Chebyshev-type inequalities for synchronous functions and extend these results to the multifunction setting. Furthermore, we show that the classical Chebyshev integral inequality is recovered as a special case of our results.

2. Preliminaries

2.1. Definition of the Operator

Definition 1.1 Let $a, b \in \mathrm{R}$ with $a < b$, $q > 0$, $n \in \mathrm{N}_0 := \{0,1,2,\dots\}$, and let $\alpha > -\frac{1}{2}$. The left hand q -generalized unified fractional remainder operator associated with the left endpoint a is given by

$$ {}_{a^+}R^{\alpha}_{q,n}(f;a,b) = \frac{q\Gamma\left(\alpha+2n+\frac{1}{2}+\frac{1}{q}\right)}{\Gamma\left(\frac{1}{q}\right)\Gamma\left(\alpha+2n+\frac{1}{2}\right)}(b-a)^{2n}\int_0^1 (1-t^q)^{\alpha+2n-\frac{1}{2}} f(a+(b-a)t)dt, \tag{2.1} $$

and the right hand operator associated with the right endpoint b by

$$ {}_{b^-}R^{\alpha}_{q,n}(f;a,b) = \frac{q\Gamma\left(\alpha+2n+\frac{1}{2}+\frac{1}{q}\right)}{\Gamma\left(\frac{1}{q}\right)\Gamma\left(\alpha+2n+\frac{1}{2}\right)}(b-a)^{2n}\int_0^1 (1-t^q)^{\alpha+2n-\frac{1}{2}} f(b-(b-a)t)dt. \tag{2.2} $$

In the proposed q-generalized unified fractional remainder operator, the parameters α, n, and q contribute to different aspects of the operator. Their respective roles can be understood by examining the kernel and the associated remainder terms.

1. The parameter α primarily influences the behavior of the kernel $(1-t^q)^{\alpha+2n-\frac{1}{2}}$. In particular, when $n = 0$, the kernel is reduced to $(1-t^q)^{\alpha-\frac{1}{2}}$. As α approaches $-\frac{1}{2}$ from above, the kernel develops a stronger singularity near $t = 1$, resulting in an increased concentration of the associated weight in that region. Conversely, larger values of α weaken this singular behavior and cause the kernel to decay more rapidly toward zero as $t \to 1^-$.
2. The parameter n is a nonnegative integer that characterizes the order of the remainder term. Its influence is reflected in both the factor $(b-a)^{2n}$ and the exponent $\alpha + 2n - \frac{1}{2}$. As n increases, the power of $(b-a)$ increases by two, and the exponent of the kernel is shifted accordingly.
3. The parameter $q > 0$, determines the form of the generalized kernel through the term $(1-t^q)$. Changing q modifies the dependence of the kernel on the variable t, thereby extending the class of kernels considered in the operator.

2.2. Well-Definedness and Normalization of the Operator

Theorem 2.1 Let $a, b \in \mathrm{R}$ with $a < b$, $q > 0$, $n \in \mathrm{N}_0$, and let $\alpha > -\frac{1}{2}$. The operators ${}_{a^+}R^{\alpha}_{q,n}$ and ${}_{b^-}R^{\alpha}_{q,n}$ satisfy the following properties:

1. Linearity. For any constants $\rho, \rho \in R$ and functions f, g

$$ {}_{a^+}R^{\alpha}_{q,n}(\rho f + \rho g;a,b) = \rho\ {}_{a^+}R^{\alpha}_{q,n}(f;a,b) + \rho\ {}_{a^+}R^{\alpha}_{q,n}(g;a,b), $$

$$ {}_{b^-}R^{\alpha}_{q,n}(\rho f + \rho g;a,b) = \rho\ {}_{b^-}R^{\alpha}_{q,n}(f;a,b) + \rho\ {}_{b^-}R^{\alpha}_{q,n}(g;a,b) $$

2. Positivity. If $f \geq 0$ on $[a,b]$, then

$$ {}_{a^+}R^{\alpha}_{q,n}(f;a,b) \geq 0, \quad {}_{b^-}R^{\alpha}_{q,n}(f;a,b) \geq 0. $$

3. Normalization. By the substitution $u = t^q$, we obtain

$$\int_0^1 (1-t^q)^{\alpha+2n-\frac{1}{2}}\,dt = \frac{1}{q}\int_0^1 u^{\frac{1}{q}-1}(1-u)^{\alpha+2n-\frac{1}{2}}du$$

$$= \frac{1}{q}\beta\left(\frac{1}{q}, \alpha+2n+\frac{1}{2}\right)$$

$$= \frac{\Gamma(\frac{1}{q})\Gamma\left(\alpha+2n+\frac{1}{2}\right)}{q\Gamma\left(\alpha+2n+\frac{1}{2}+\frac{1}{q}\right)}. \tag{2.3}$$

Thus, the normalized kernel

$$w_{\alpha,q,n}(t) = \frac{q\Gamma\left(\alpha+2n+\frac{1}{2}+\frac{1}{q}\right)}{\Gamma(\frac{1}{q})\Gamma\left(\alpha+2n+\frac{1}{2}\right)}(1-t^q)^{\alpha+2n-\frac{1}{2}}, \qquad 0 \le t \le 1, \tag{2.4}$$

is nonnegative and satisfies

$$\int_0^1 w_{\alpha,q,n}(t)dt = 1.$$

4. For every $f \in C([a,b])$, the operator norm satisfies

$$\sup_{\|f\|_\infty \le 1} \left|\ {}_{a^+}R^{\alpha}_{q,n}(f;a,b)\right| = \sup_{\|f\|_\infty \le 1} \left|\ {}_{b^-}R^{\alpha}_{q,n}(f;a,b)\right| = (b-a)^{2n}$$

Equivalently

$${}_{a^+}\mathrm{A}^{\alpha}_{q,n}(f;a,b) = \frac{1}{(b-a)^{2n}}\ {}_{a^+}R^{\alpha}_{q,n}(f;a,b), \quad {}_{b^-}\mathrm{A}^{\alpha}_{q,n}(f;a,b) = \frac{1}{(b-a)^{2n}}\ {}_{b^-}R^{\alpha}_{q,n}(f;a,b).$$

are bounded linear functionals on $C([a,b])$.

5. For every $f \in C([a,b])$, the normalized operators ${}_{a^+}\mathrm{A}^{\alpha}_{q,n}$ and ${}_{b^-}\mathrm{A}^{\alpha}_{q,n}$ recover the endpoint values on shrinking intervals. By the dominated convergence theorem, and the continuity of f on the compact interval and the integrability of the kernel, we obtain

$$\lim_{h\to 0^+}\ {}_{a^+}\mathrm{A}^{\alpha}_{q,n}(f;a,a+h) = \lim_{h\to 0^+}\int_0^1 w_{\alpha,q,n}(t)f(a+ht)dt$$

$$= \int_0^1 \lim_{h\to 0^+} w_{\alpha,q,n}(t)f(a+ht)dt$$

$$= f(a)\int_0^1 w_{\alpha,q,n}(t)dt$$

$$= f(a),$$

$$\lim_{h\to 0^+}\ {}_{b^-}\mathrm{A}^{\alpha}_{q,n}(f;b-h,b) = \lim_{h\to 0^+}\int_0^1 w_{\alpha,q,n}(t)f(b-ht)dt$$

$$= \int_0^1 \lim_{h\to 0^+} w_{\alpha,q,n}(t)f(b-ht)dt$$

$$= f(b)\int_0^1 w_{\alpha,q,n}(t)dt$$

$$= f(b).$$

Hence, ${}_{a^+}\mathrm{A}^{\alpha}_{q,n}$, and ${}_{b^-}\mathrm{A}^{\alpha}_{q,n}$ may be regarded as a fractional weighted averaging operators.

Remark 1 The relation between the left-hand and right-hand normalized operators follows directly from their definitions by a change of variable.

1. Using the substitution $t = 1 - s$ in the right-hand operator ${}_{b}\text{-A}_{q,n}^{\alpha}$, we obtain

$$
\begin{aligned}
{}_{b}\text{-A}_{q,n}^{\alpha}(f;a,b) &= \int_0^1 w_{\alpha,q,n}(t) f(b-(b-a)t)dt \\
&= \int_0^1 w_{\alpha,q,n}(1-s) f(a+(b-a)s)ds.
\end{aligned}
$$

2. For $\alpha = \frac{1}{2}$, $q = 2$, and $n = 0$, the normalized operators are reduced to the standard integral mean

$$
\begin{aligned}
{}_{a}\text{+A}_{2,0}^{\frac{1}{2}}(f;a,b) &= \frac{2\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma(1)} \int_0^1 f(a+(b-a)t)dt \\
&= \frac{2\Gamma\left(1+\frac{1}{2}\right)}{(b-a)\Gamma\left(\frac{1}{2}\right)} \int_a^b f(s)ds \\
&= \frac{1}{b-a} \int_a^b f(s)ds. \qquad (2.5)
\end{aligned}
$$

The same result is obtained for the right hand side normalized operator ${}_{b}\text{-A}_{2,0}^{\frac{1}{2}}(f;a,b)$.

3. Chebyshev-type Inequality for Unified Fractional Remainder Operator

In this section, we introduce a fractional Chebyshev functional T_2 associated with the normalized unified fractional remainder operator. More precisely, we establish a double-integral representation for this functional and use the synchronous property of the functions together with the positivity of the fractional kernel to obtain a fractional analogue of the classical Chebyshev inequality.

3.1. Chebyshev Functional

Theorem 1.1 Let $a, b \in \mathrm{R}$ with $a < b$, $q > 0$, $n \in \mathrm{N}_0$, $\alpha > -\frac{1}{2}$, and let $f, g: [a,b] \to \mathrm{R}$ be continuous functions. We define the fractional Chebyshev functional associated with ${}_{a}\text{+A}_{q,n}^{\alpha}$ by

$$
\mathrm{T}_2(f,g;a,b) = {}_{a}\text{+A}_{q,n}^{\alpha}(fg;a,b) - {}_{a}\text{+A}_{q,n}^{\alpha}(f;a,b)\ {}_{a}\text{+A}_{q,n}^{\alpha}(g;a,b). \qquad (3.1)
$$

Equivalently,

$$
\mathrm{T}_2(f,g;a,b) = \frac{1}{2}\int_0^1 \int_0^1 (F(t)-F(s))\,(G(t)-G(s)) w_{\alpha,q,n}(t) w_{\alpha,q,n}(s) ds dt. \qquad (3.2)
$$

Proof. For convenience, set

$$
F(t) = f(a+(b-a)t), \qquad G(t) = g(a+(b-a)t),
$$

and define

$$
\Phi(t,s) = (F(t)-F(s))(G(t)-G(s)), \quad (t,s) \in [0,1] \times [0,1].
$$

Since $f, g \in C([a,b])$, the functions F, G, and FG are bounded on $[0,1]$. Consequently, multiplying $\Phi(t,s)$ by the product of the normalized kernels $w_{\alpha,q,n}(t) w_{\alpha,q,n}(s)$, and double integration over the square $[0,1] \times [0,1]$ gives

$$
\begin{aligned}
\int_0^1 \int_0^1 |\Phi(t,s)|\, w_{\alpha,q,n}(t) w_{\alpha,q,n}(s) ds dt &\le 4\|F\|_\infty \|G\|_\infty \left(\int_0^1 w_{\alpha,q,n}(t)dt\right)^2 \\
&\le 4\|f\|_\infty \|g\|_\infty < \infty.
\end{aligned}
$$

Thus, the integrand is absolutely integrable on $[0,1] \times [0,1]$ with respect to Lebesgue measure. Therefore, Fubini's theorem allows to interchange the order of integration.

$$\int_0^1\int_0^1 (F(t)-F(s))\,(G(t)-G(s))w_{\alpha,q,n}(t)w_{\alpha,q,n}(s)dsdt$$

$$= \int_0^1\int_0^1 (F(t)G(t)-F(t)G(s)-F(s)G(t)+F(s)G(s))\,w_{\alpha,q,n}(t)w_{\alpha,q,n}(s)dsdt$$

$$= \int_0^1 w_{\alpha,q,n}(t)\,F(t)G(t)\underbrace{\left(\int_0^1 w_{\alpha,q,n}(s)ds\right)}_{=1}dt - \int_0^1 F(t)w_{\alpha,q,n}\,(t)\left(\int_0^1 w_{\alpha,q,n}(s)G(s)ds\right)dt$$

$$-\int_0^1 w_{\alpha,q,n}(t)G(t)\left(\int_0^1 w_{\alpha,q,n}(s)F(s)ds\right)dt + \underbrace{\left(\int_0^1 w_{\alpha,q,n}(t)dt\right)}_{=1}\left(\int_0^1 w_{\alpha,q,n}(s)F(s)G(s)ds\right)$$

$$= \int_0^1 w_{\alpha,q,n}(t)F(t)G(t)dt - 2\left(\int_0^1 F(t)w_{\alpha,q,n}(t)dt\right)\left(\int_0^1 w_{\alpha,q,n}(s)G(s)ds\right)$$

$$+\int_0^1 w_{\alpha,q,n}(s)F(s)G(s)ds$$

$$= 2\ {}_{a^+}\mathrm{A}^{\alpha}_{q,n}(fg;a,b) - 2\ {}_{a^+}\mathrm{A}^{\alpha}_{q,n}(f;a,b)\ {}_{a^+}\mathrm{A}^{\alpha}_{q,n}(g;a,b)$$

Averaging the two sides gives

$$\mathrm{T}_2(f,g;a,b) = \tfrac{1}{2}\int_0^1\int_0^1 (F(t)-F(s))(G(t)-G(s))w_{\alpha,q,n}\,(t)w_{\alpha,q,n}(s)dsdt$$

$$= {}_{a^+}\mathrm{A}^{\alpha}_{q,n}(fg;a,b) - {}_{a^+}\mathrm{A}^{\alpha}_{q,n}(f;a,b)\ {}_{a^+}\mathrm{A}^{\alpha}_{q,n}(g;a,b).$$

3.2. Chebyshev Inequality for Synchronous Functions

Recall that two functions $f,g:[a,b]\to R$ are said to be synchronous if

$$(f(x)-f(y))(g(x)-g(y))\geq 0,$$

for all $x,y\in[a,b]$.

Theorem 2.1 Let $a,b\in\mathrm{R}$ with $a<b$, $q>0$, $n\in\mathrm{N}_0$, $\alpha>-\frac{1}{2}$, and let $f,g:[a,b]\to\mathrm{R}$ be continuous synchronous functions. Then

$${}_{a^+}\mathrm{A}^{\alpha}_{q,n}(fg;a,b)\geq\ {}_{a^+}\mathrm{A}^{\alpha}_{q,n}(f;a,b)\ {}_{a^+}\mathrm{A}^{\alpha}_{q,n}(g;a,b). \tag{3.3}$$

Equivalently,

$${}_{a^+}R^{\alpha}_{q,n}(fg;a,b)\geq\frac{1}{(b-a)^{2n}}\ {}_{a^+}R^{\alpha}_{q,n}(f;a,b)\ {}_{a^+}R^{\alpha}_{q,n}(g;a,b). \tag{3.4}$$

Proof. Since f and g are synchronous for every $x,y\in[a,b]$, we have

$$(f(x)-f(y))(g(x)-g(y))\geq 0.$$

Let

$$x=a+(b-a)t,\quad y=a+(b-a)s,\qquad s,t\in[0,1].$$

Hence

$$(f(a+(b-a)t)-f(a+(b-a)s))(g(a+(b-a)t)-g(a+(b-a)s))\geq 0.$$

Furthermore, since $w_{\alpha,q,n}(t)$ is nonnegative kernel. Therefore,

$$(f(a+(b-a)t)-f(a+(b-a)s))(g(a+(b-a)t)-g(a+(b-a)s))w_{\alpha,n}(t)w_{\alpha,n}(s)\geq 0.$$

Averaging the inequality, and double integration over the square $[0,1] \times [0,1]$ gives

$$\frac{1}{2}\int_0^1 \int_0^1 (f(a+(b-a)t) - f(a+(b-a)s))(g(a+(b-a)t) - g(a+(b-a)s))w_{\alpha,q,n}(t)w_{\alpha,q,n}(s)dsdt \geq 0.$$

By (3.2), the left-hand side is precisely $T_2(f,g;a,b)$. Hence

$$\mathrm{T}_2(f,g;a,b) \geq 0,$$

which proves

$${}_{a^+}\mathrm{A}^{\alpha}_{q,n}(fg;a,b) \geq_{a^+} \mathrm{A}^{\alpha}_{q,n}(f;a,b)\,{}_{a^+}\mathrm{A}^{\alpha}_{q,n}(g;a,b).$$

Finally, we obtain

$${}_{a^+}R^{\alpha}_{q,n}(fg;a,b) \geq \frac{1}{(b-a)^{2n}}\ {}_{a^+}R^{\alpha}_{q,n}(f;a,b)\,{}_{a^+}R^{\alpha}_{q,n}(g;a,b).$$

Remark 2 For $\alpha = \frac{1}{2}$, $q = 2$, and $n = 0$, the Chebyshev integral inequality (3.4) is reduced to the classical Chebyshev inequality (1.4)

$${}_{a^+}\mathrm{A}^{\frac{1}{2}}_{2,0}(fg;a,b) = \frac{1}{b-a}\int_a^b f(t)g(t)dt$$

$$\geq_{a^+} \mathrm{A}^{\frac{1}{2}}_{2,0}(f;a,b)\ {}_{a^+}\mathrm{A}^{\frac{1}{2}}_{2,0}(g;a,b) = \frac{1}{(b-a)^2}\left(\int_a^b f(t)dt\right)\left(\int_a^b g(t)dt\right).$$

Example 1 Consider the following parameters: $-\frac{1}{2} < \alpha \leq 5$, $\frac{1}{2} \leq q \leq 5$, $[a,b] = [1,3]$, and the synchronous functions $f(t) = t^2$, $g(t) = t^3$. For each $t \in [a,b]$, we can numerically evaluate the Chebyshev inequality (3.4) (Fig. **1**).

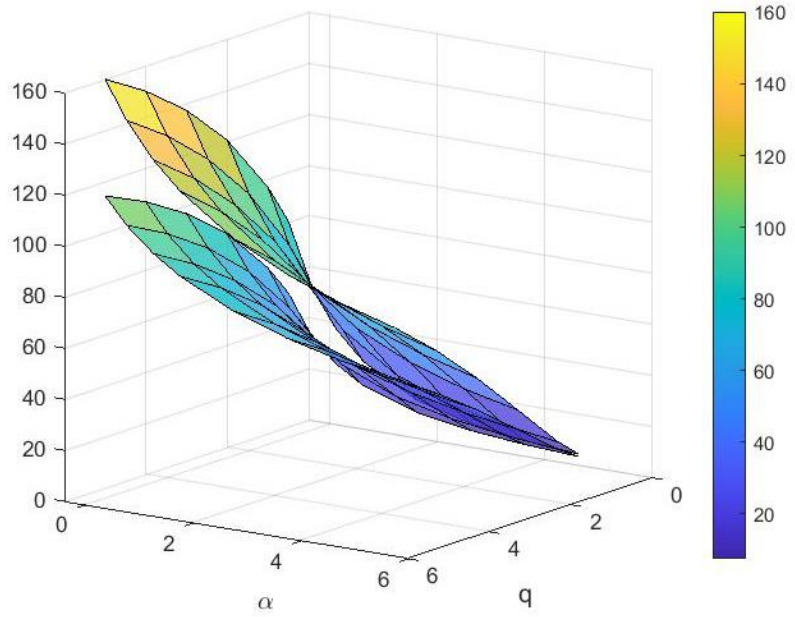


Figure 1: Numerical illustration of the Chebyshev integral inequality (3.4) for the synchronous functions $f(t) = t^2$, $g(t) = t^3$ on the interval $[a,b] = [1,3]$, with parameters $-\frac{1}{2} < \alpha \leq 5$, $\frac{1}{2} \leq q \leq 5$, and $n = 1$.

Note that LHS (upper surface) = ${}_{a^+}A^{\alpha}_{q,n}(fg;a,b)$, RHS (lower surface) = ${}_{a^+}A^{\alpha}_{q,n}(fg;a,b)\ {}_{a^+}A^{\alpha}_{q,n}(g;a,b)$. For the selected values of $\alpha \in \{-0.4,0,0.5,1\}$, $q \in \{0.5,1,2,3,4,5\}$, the Chebyshev inequality (3.4) was numerically verified (Table **1**).

Table 1: Numerical verification of the Chebyshev inequality (3.4).

α	q	LHS	RHS	Difference	α	q	LHS	RHS	Difference
-0.4	0.5	80.6637	52.6539	28.0098	-0.4	3	137.3530	96.8815	40.4715
-0.4	1	96.5945	64.9850	31.6095	-0.4	4	150.1318	106.7665	43.3654
-0.4	2	120.3700	83.6136	36.7564	-0.4	5	160.1127	114.4026	45.7101
0	0.5	58.6074	38.4661	20.1413	0	3	119.6337	84.7497	34.8840
0	1	75.1875	50.8126	24.3749	0	4	134.0367	95.7175	38.3192
0	2	100.8094	70.3355	30.4739	0	5	145.4116	104.3085	41.1031
0.5	0.5	41.3200	27.7301	13.5899	0.5	3	103.5488	73.8720	29.6768
0.5	1	57.4286	39.3600	18.0686	0.5	4	119.1414	85.5898	33.5516
0.5	2	83.5952	58.8469	24.7483	0.5	5	131.6252	94.9162	36.7090
1	0.5	30.5842	21.1996	9.3846	1	3	91.7073	65.9285	25.7788
1	1	45.6158	31.8669	13.7489	1	4	107.9456	78.0252	29.9204
1	2	71.3465	50.7622	20.5842	1	5	121.1164	87.7930	33.3234

4. Multifunction Chebyshev-type Inequality for Unified Fractional Remainder Operator

4.1. Chebyshev Functional

Theorem 1.1 Let $a, b \in \mathrm{R}$ with $a < b$, $q > 0$, $m \geq 2$, $n \in \mathrm{N}_0$, $\alpha > -\frac{1}{2}$, and let $f_1, f_2, \dots, f_m : [a, b] \to [0, \infty)$ be continuous and nondecreasing functions. We define the multifunction Chebyshev functional associated with ${}_{a^+}\mathrm{A}^{\alpha}_{q,n}$ by

$$\mathrm{T}_m(f_1, f_2, \dots, f_m; a, b) = {}_{a^+}\mathrm{A}^{\alpha}_{q,n}\left(\prod_{k=1}^{m} f_k; a, b\right) - \prod_{k=1}^{m} {}_{a^+}\mathrm{A}^{\alpha}_{q,n}(f_k; a, b). \tag{4.1}$$

Equivalently,

$$\mathrm{T}_m(f_1, \dots, f_m; a, b) = \frac{1}{2}\sum_{k=2}^{m}\left(\prod_{j=k+1}^{m} {}_{a^+}\mathrm{A}^{\alpha}_{q,n}(f_j)\right)\int_0^1\int_0^1\left[\prod_{i=1}^{k-1} f_i(a + (b-a)t) - \prod_{i=1}^{k-1} f_i(a + (b-a)s)\right]$$
$$\times\left[f_k(a + (b-a)t) - f_k(a + (b-a)s)\right] w_{\alpha,q,n}(t) w_{\alpha,q,n}(s)\, dt\, ds. \tag{4.2}$$

For convenience, we adopt the convention that an empty product is equal to 1, in particular

$$\prod_{j=m+1}^{m} {}_{a^+}\mathrm{A}^{\alpha}_{q,n}(f_j; a, b) = 1.$$

Proof. Let $m \geq 2$ be a natural number, for $k = 1, \dots, m$, we define

$$F_k(x) = \prod_{i=1}^{k} f_i(x).$$

Thus,

$$\begin{aligned}
\mathrm{T}_m(f_1, \dots, f_m) &= {}_{a^+}\mathrm{A}^{\alpha}_{q,n}\left(\prod_{i=1}^{m} f_i\right) - \prod_{i=1}^{m} {}_{a^+}\mathrm{A}^{\alpha}_{q,n}(f_i) \\
&= {}_{a^+}\mathrm{A}^{\alpha}_{q,n}(F_{m-1} f_m) - {}_{a^+}\mathrm{A}^{\alpha}_{q,n}(F_{m-1})\ {}_{a^+}\mathrm{A}^{\alpha}_{q,n}(f_m) \\
&\quad + {}_{a^+}\mathrm{A}^{\alpha}_{q,n}(f_m)\left[{}_{a^+}\mathrm{A}^{\alpha}_{q,n}(F_{m-1}) - \prod_{i=1}^{m-1} {}_{a^+}\mathrm{A}^{\alpha}_{q,n}(f_i)\right] \\
&= \mathrm{T}_2(F_{m-1}, f_m) + {}_{a^+}\mathrm{A}^{\alpha}_{q,n}(f_m)\,\mathrm{T}_{m-1}(f_1, \dots, f_{m-1})
\end{aligned}$$

Iterating this identity yields

$$= \sum_{k=2}^{m}\left(\prod_{j=k+1}^{m} {}_{a^+}\mathrm{A}_{q,n}^{\alpha}(f_j)\right)\mathrm{T}_2(F_{k-1}, f_k)$$

Using the two function representation (3.2), we obtain

$$= \frac{1}{2}\sum_{k=2}^{m}\left(\prod_{j=k+1}^{m} {}_{a^+}\mathrm{A}_{q,n}^{\alpha}(f_j)\right)\int_0^1\int_0^1\left[\prod_{i=1}^{k-1} f_i(a+(b-a)t) - \prod_{i=1}^{k-1} f_i(a+(b-a)s)\right]$$

$$\times [f_k(a+(b-a)t) - f_k(a+(b-a)s)]w_{\alpha,q,n}(t)w_{\alpha,q,n}(s)dtds.$$

Remark 3 We note that the nonnegativity and monotonicity assumptions imposed on $f_1, \dots, f_m$ are sufficient conditions for the validity of the multifunction Chebyshev inequality. Nonnegativity ensures that the partial products remain nondecreasing and that the operator means in the recursive representation are nonnegative, while monotonicity guarantees the synchrony required to apply the two-function Chebyshev inequality.

4.2. Multifunction Chebyshev Inequality

Theorem 2.1 Let $a, b \in \mathrm{R}$ with $a < b$, $q > 0$, $n \in \mathrm{N}_0$, $\alpha > -\frac{1}{2}$, and let $f_1, f_2, \dots, f_m: [a, b] \to [0, \infty)$ be continuous and nondecreasing functions. Then

$$ {}_{a^+}\mathrm{A}_{q,n}^{\alpha}\left(\prod_{i=1}^{m} f_i\right) \geq \prod_{i=1}^{m} {}_{a^+}\mathrm{A}_{q,n}^{\alpha}(f_i). \tag{4.3}$$

Equivalently

$$ {}_{a^+}R_{q,n}^{\alpha}\left(\prod_{i=1}^{m} f_i; a, b\right) \geq \frac{1}{(b-a)^{2n(m-1)}}\prod_{i=1}^{m} {}_{a^+}R_{q,n}^{\alpha}(f_i; a, b). \tag{4.4}$$

Proof. Since all f_i are nonnegative and nondecreasing on $[a, b]$, then F_{k-1} and f_k are both nondecreasing,

$$[F_{k-1}(x) - F_{k-1}(y)][f_k(x) - f_k(y)] \geq 0, \qquad x, y \in [a, b].$$

Moreover,

$$ {}_{a^+}\mathrm{A}_{q,n}^{\alpha}(f_i) \geq 0.$$

Therefore, every term in the above sum is nonnegative. Thus,

$$\mathrm{T}_m(f_1, \dots, f_m) \geq 0,$$

which proves that

$$ {}_{a^+}\mathrm{A}_{q,n}^{\alpha}\left(\prod_{i=1}^{m} f_i\right) \geq \prod_{i=1}^{m} {}_{a^+}\mathrm{A}_{q,n}^{\alpha}(f_i).$$

Finally, since

$$ {}_{a^+}R_{q,n}^{\alpha}(f; a, b) = (b-a)^{2n} {}_{a^+}\mathrm{A}_{q,n}^{\alpha}(f; a, b),$$

we obtain

$$ {}_{a^+}R_{q,n}^{\alpha}\left(\prod_{i=1}^{m} f_i; a, b\right) \geq \frac{1}{(b-a)^{2n(m-1)}}\prod_{i=1}^{m} {}_{a^+}R_{q,n}^{\alpha}(f_i; a, b).$$

The proof is complete.

Remark 4 For $\alpha = \frac{1}{2}$, $q = 2$, and $n = 0$, the Chebyshev integral inequality (4.4) is reduced to the classical Chebyshev inequality (1.8)

$$\frac{1}{b-a}\int_a^b \prod_{i=1}^{m} f_i(t)dt \geq \prod_{i=1}^{m}\left(\frac{1}{b-a}\int_a^b f_i(t)dt\right).$$

5. Conclusion

Motivated by the close relationship between weighted integral means and Chebyshev's inequality, in this paper we introduced a new fractional Chebyshev inequality. By using the double integration and the synchronous property of the functions together with the positivity of the fractional kernel, we obtained a fractional analogue of the classical two-functions and multifunction Chebyshev inequalities. Furthermore, we verified that the classical versions are recovered for the appropriate choice of the parameters. The results obtained provide a unified fractional framework for Chebyshev-type inequalities and may serve as a basis for further developments involving variable-order fractional operators, two-sided fractional operators, and inequalities for functions of several variables.

Appendix A: MATLAB Code for Numerical Verification

The following MATLAB program numerically evaluates both sides of the fractional Chebyshev inequality for the selected values of α and q, and plots the results.

```
% Parameters

a = 1;
b = 3;
n = 1;

% Synchronous functions

f = @(x) x.^2;
g = @(x) x.^3;
fg = @(x) x.^5;

% Selected parameter values

alpha_values = [-0.4, 0, 0.5, 1, 2, 3, 4, 5];
q_values = [0.5, 1, 2, 3, 4, 5];

% Initialize matrices

LHS = zeros(length(alpha_values), length(q_values));
RHS = zeros(length(alpha_values), length(q_values));

% Evaluate both sides of the inequality

for i = 1:length(alpha_values)
alpha = alpha_values(i);
for j = 1:length(q_values)
q = q_values(j);
Rf = R_operator(f, a, b, alpha, q, n);
Rg = R_operator(g, a, b, alpha, q, n);
Rfg = R_operator(fg, a, b, alpha, q, n);
LHS(i,j) = Rfg;
RHS(i,j) = (b-a)^(-2*n) * Rf * Rg;
end
end
```

```
% Compute the difference

Difference = LHS - RHS;
disp('LHS:');
disp(LHS);
disp('RHS:');
disp(RHS);
disp('LHS - RHS:');
disp(Difference);

% Plot both sides

[Q, Alpha] = meshgrid(q_values, alpha_values);
figure;
surf(Q, Alpha, LHS);
hold on;
surf(Q, Alpha, RHS);
xlabel('q');
ylabel('\alpha');
zlabel('Operator value');
legend('LHS (upper surface) = A(fg)','RHS (lower surface) = A(f)*A(g)','Location','best');
colorbar;
grid on;
view(135, 30);
hold off;

% Fractional remainder operator

function R = R_operator(fun, a, b, alpha, q, n)
C = q * gamma(alpha + 2*n + 1/2 + 1/q) / (gamma(1/q) *gamma(alpha + 2*n + 1/2));
integrand = @(t) (1-t.^q).^(alpha + 2*n - 1/2) .* fun(a + (b-a).*t);
I = integral(integrand, 0, 1, 'RelTol', 1e-10, 'AbsTol', 1e-12);
R = C * (b-a)^(2*n) * I;
end
```

Conflict of Interest

The authors declare that they have no conflict of interest regarding the publication of this paper.

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