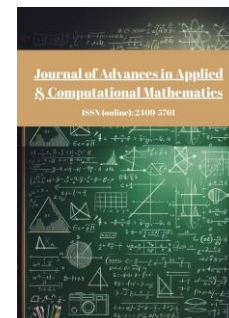




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Positive Solutions of a Riemann-Liouville Fractional Dirichlet Boundary Value Problem: Existence, Uniqueness, and Parameter Dependence

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ABSTRACT

In this article, we discuss the parameter dependence of positive solutions to fractional differential equations with Dirichlet boundary value conditions. Through fixed point theorems about increasing operators and decreasing operators, the existence and uniqueness of positive solutions is proved for each given fixed positive parameter. We analyze the monotonicity, continuity and limit behavior of solutions with respect to the parameter, and an iterative sequence is established to approximate this unique solution. Finally, we give two examples which prove the applicability and validity of the main conclusion.

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1. Introduction

Owing to excellent performance in describing the dynamics of complex systems, fractional-order models have been increasingly applied, and a number of integer-order models have been transformed into fractional-order models for research [1-6]. For instance, the authors [3] incorporated fractional derivatives into the compartmental model of the Ebola epidemic. In [4], the authors verified that the fractional-order compartmental model is highly consistent with the data published by the World Health Organization.

Nowadays, fractional differential equation (FDE for short) has been as a good tool for the modeling of different phenomena. Therefore, fractional boundary value problems (FBVPs for short) have been a hot research topic for decades due to their extensive applications in the scientific field. One of the core research contents of boundary value problems (BVPs for short) is the existence results of solutions [7-22]. Many researchers have focused on investigating BVPs composed of specific boundary conditions (BCs for short) and FDEs, [23-39] and the related references cited therein.

In [9], Cui studied the following BVP for a FDE

$$\begin{cases} (-D_{0+}^p Y(t) - p(t)f(t, Y(t)) + q(t) = 0, & t \in (0,1), \\ Y(0) = Y'(0) = 0, & Y(1) = 0, \end{cases}$$

where D_{0+}^p is the standard Riemann-Liouville derivative (RLD for short) and $2 < p \leq 3$ is a real number. Under the assumption that $f(t, Y)$ is a Lipschitz continuous function, the author studied the uniqueness and existence of solutions for the above FDE by using fixed point theorems of u_0 -positive operators.

In [10], Zhang and Zhong studied a FBVP

$$\begin{cases} D_{0+}^\alpha Y(t) + f(t, Y(t)) = 0, & 0 < t < 1, \\ Y(0) = Y'(0) = \dots = Y^{(n-2)}(0) = 0, & D_{0+}^\beta Y(1) = \lambda \int_0^\eta h(t) D_{0+}^\beta Y(t) dt, \end{cases}$$

where $D_{0+}^\alpha, D_{0+}^\beta$ are the RLDs of orders α, β respectively, and satisfy $n-1 < \alpha \leq n$ ($n \geq 3$), $\beta \geq 1$, $\alpha - \beta - 1 > 0$, $0 < \eta \leq 1$, and $0 \leq \lambda \int_0^\eta h(t) t^{\alpha-\beta-1} dt < 1$. The authors used Leggett-Williams fixed point theorem and Krasnosel'skii fixed point theorem as core tools to prove the existence of triple positive solutions.

In [11], the authors considered a FBVP

$$\begin{cases} D_{0+}^\beta Y(t) + \lambda g(t, Y(t)) = 0, & 0 < t < 1, \quad n-1 < \beta < n, \quad n \geq 3, \\ Y^{(j)}(0) = 0, j = 0, 1, \dots, n-2, & D_{0+}^p Y(1) = \sum_{i=1}^m \eta_i D_{0+}^q Y(t_i), \end{cases}$$

where D_{0+}^β denotes the RLD of order $\beta \in (n-1, n]$ ($n \in \mathbb{N}$ and $n \geq 3$), $0 < t_1 < \dots < t_m < 1$ ($m \in \mathbb{N}$), $p \in [1, n-2]$, $q \in [0, p]$, $\eta_i \geq 0$ ($i = 1, \dots, m$), the parameter $\lambda > 0$, and $g(t, Y)$ is a continuous function and may change sign and may be singular at $t = 0$ or $t = 1$. The authors used the Guo-Krasnosel'skii fixed point theorem on cones as the core tool to prove the existence of positive solutions by constructing operators.

In [16], Wang studied the following FBVP

$$\begin{cases} (-D_{0+}^\alpha Y(t) + \lambda a(t)Y(t) = y(t), & t \in (0,1), \quad \alpha \in (2,3], \\ Y(0) = Y(1) = Y'(0) = 0, \end{cases}$$

where D_{0+}^α is a RLD of order α , the parameter $\lambda > 0$, $a \in L^1[0,1]$. The author derived some new properties of the corresponding Green function, and got the multiplicity and existence of positive solutions by utilizing the fixed point index theory.

In [17], the author discussed the following FBVP

$$\begin{cases} -D_{0+}^{\alpha}Y(t) + a(t)Y(t) = y(t), & t \in (0,1), \alpha \in (1,2), \\ Y(1) = 0, \quad Y(0) = 0, \end{cases}$$

where D_{0+}^{α} is the RLD, and $a \in L^1[0,1]$. The author obtained some positive properties of the Green's function for this problem under certain conditions. In addition, the paper considered a semi-positone FDE: $-D_{0+}^{\alpha}Y(t) = \mu F(t, Y(t))$, $t \in (0,1)$, $\alpha \in (1,2)$, and obtained the existence of positive solutions under a more general condition: $F(t, Y) \geq -b(t)Y - e(t)$, where $b(t), e(t) \in L^1[0,1]$ are non-negative functions. However, the uniqueness of positive solutions was not considered.

In [18] and [19], the authors studied the FBVP

$$\begin{cases} -D_{0+}^{\gamma}Y(t) + a(t)Y(t) = g(t, Y(t)), & 0 < t < 1, \gamma \in (2,3), \\ Y'(1) = 0, \quad Y(0) = Y'(0) = 0, \end{cases}$$

where D_{0+}^{γ} is a RLD of order γ , $a \in C[0,1]$ with $\bar{a} = \max\{|a(t)|: t \in [0,1]\} < \frac{(\gamma-1)\Gamma(\gamma+1)}{\gamma^{\gamma+1}}$. The authors deduced some positive properties of the Green's function for the above FBVP. They proved the uniqueness of solutions by using mixed monotone operator theory and the Schauder fixed point theorem.

Inspired by the above work, we study the following FDE with Dirichlet BCs

$$\begin{cases} -D_{0+}^{\alpha}Y(t) + a(t)Y(t) = \lambda g(t, Y(t)), & 0 < t < 1, \\ Y(0) = 0, \quad Y(1) = 0, \end{cases} \quad (1.1)$$

where D_{0+}^{α} is the RLD, $\alpha \in (1,2)$, $\lambda > 0$ is a parameter and $a \in L^1[0,1]$. We obtain the existence and uniqueness of positive solutions through fixed point theorems of monotonically increasing and decreasing operators, and analysis the relationship between the solution and the parameter. In [41, 42], we gave the existence, uniqueness and the parameter dependence of positive fixed points for increasing operators. But for decreasing operators, the parameter dependence has not been obtained directly. To fill the gap, we first discuss the parameter dependence of positive fixed points for decreasing operators. The structure of this paper is as follows, in the second part we give some basic definitions and related properties of the Green function, and give some contents of fixed point theorems. The third part gives the relevant nature of positive solutions to the problem (1.1). Finally, two examples are given to demonstrate the applicability and validity of the main results.

2. Preliminaries and previous results

Definition 2.1 ([40]). Let $\Omega = 0, b(0 < b < \infty)$. For a function $f: \Omega \rightarrow R$, the Riemann-Liouville fractional integral of order $\rho > 0$ is defined as

$$I_{0+}^{\rho}f(x) = \frac{1}{\Gamma(\rho)} \int_0^x \frac{f(t)}{(x-t)^{1-\rho}} dt \quad (x > 0; \rho > 0).$$

Here $\Gamma(\rho)$ is the Gamma function defined by

$$\Gamma(\rho) = \int_0^{+\infty} t^{\rho-1} e^{-t} dt, \rho > 0.$$

Definition 2.2 ([40]). The RLD of order $s \in (1,2)$ of a function $f(x)$ is given by

$$D_{0+}^s f(x) = \frac{1}{\Gamma(2-s)} \left(\frac{d}{dx} \right)^2 \int_0^x (x-t)^{2-s-1} f(t) dt.$$

Lemma 2.1 ([17]). Let $\alpha \in (1, 2)$, $a \in L^1[0, 1]$ and suppose that

- (i) $\Lambda_1 = \int_0^1 |a(t)|[t(1-t)]^{\alpha-1} dt < \Gamma(\alpha)$;
- (ii) $\Lambda_2 = \int_0^1 |a(t)|[t(1-t)]^{\alpha-2} dt < +\infty$;
- (iii) $\Lambda = \frac{\Lambda_2}{(\alpha-1)(\Gamma(\alpha)-\Lambda_1)} < 1$.

Then, for any $y \in L^1[0, 1]$, the FBVP

$$\begin{cases} (-D_{0+}^\alpha Y(t) + a(t)Y(t) = y(t), & 0 < t < 1, \\ Y(0) = 0, & Y(1) = 0 \end{cases}$$

admits a unique solution given by

$$Y(t) = \int_0^1 t^{\alpha-2} k_\alpha(t, s) y(s) ds,$$

where $k_\alpha(t, s) = \sum_{n=0}^{+\infty} (-1)^n k_n(t, s)$, and the kernel $k_n(t, s)$ is defined recursively as

$$\begin{aligned} k_n(t, s) &= \int_0^1 a(\eta) \eta^{\alpha-2} k_0(t, \eta) k_{n-1}(\eta, s) d\eta \quad (n = 1, 2, \dots), \\ k_0(t, s) &= \frac{t^{2-\alpha}}{\Gamma(\alpha)} \begin{cases} [s(1-t)]^{\alpha-1}, & 0 \leq s \leq t \leq 1, \\ [s(1-t)]^{\alpha-1} - (s-t)^{\alpha-1}, & 0 \leq t \leq s \leq 1. \end{cases} \end{aligned}$$

Lemma 2.2 ([17]). From Lemma 3 in [17], it is known that $k_\alpha(t, s)$ is a uniformly convergent function series on $[0, 1] \times [0, 1]$. Moreover, $k_\alpha(t, s)$ satisfies the following properties:

- (i) $k_\alpha(t, s) \geq \frac{(1-\Lambda)p(t)q(s)}{\Gamma(\alpha-1)}, \quad \forall s, t \in [0, 1];$
- (ii) $k_\alpha(t, s) \leq \frac{(1+\Lambda)p(t)q(s)}{\Gamma(\alpha)p(s)}, \quad \forall s, t \in (0, 1);$
- (iii) $k_\alpha(t, s) \leq \frac{(1+\Lambda)q(s)}{\Gamma(\alpha)}, \quad \forall s, t \in [0, 1];$

where $p(t) = p(1-t) = t(1-t)$, $q(t) = t(1-t)^{\alpha-1}$.

It is worth mentioning, for $s, t \in (0, 1)$, from (i)(ii) in Lemma 2.2, we know that $\frac{(1-\Lambda)}{\Gamma(\alpha-1)} \leq \frac{(1+\Lambda)}{\Gamma(\alpha)p(s)}$.

Let $(E, \|\cdot\|)$ be a real Banach space, $K \subset E$ is a cone. The space E is partially ordered by the cone K , i.e., $x \leq y$ if and only if $y - x \in K$. If $x \leq y$ and $x \neq y$, we denote $x < y$ or $y > x$. If there is $N > 0$ such that for $x, y \in E$, $\theta \leq x \leq y$ yields $\|x\| \leq N\|y\|$, then K is called normal and N is called the normality constant of K . An operator $A: E \rightarrow E$ is increasing if $x \leq y$ yields $Ax \leq Ay$. Similarly, one can define decreasing operators. For $x, y \in E$, the notation $x: y$ means that there exist $\lambda, \mu > 0$ such that $\lambda x \leq y \leq \mu x$. Given $h > \theta$, denote a set $K_h = \{x \in E | x: h\}$, then $K_h \subset K$.

Lemma 2.3 ([41, 42]). Let K be a normal cone in a real Banach space E , $h > \theta$, and let $A: K \rightarrow K$ be an increasing operator which satisfies:

1. There is $h_0 \in K_h$ such that $Ah_0 \in K_h$;
2. For any $x \in K$ and $t \in (0, 1)$, there exists $\phi(t) \in (t, 1)$ such that $A(tx) \geq \phi(t)Ax$.

Then

1. The equation $Ax = x$ has a unique solution x^* in K_h ;
2. Giving an initial value $x_0 \in K_h$, if $x_n = Ax_{n-1}$, $n = 1, 2, \dots$, then $x_n \rightarrow x^*$ as $n \rightarrow \infty$.

Lemma 2.4 ([41,42]). Under the conditions of Lemma 2.3 and let $x_\lambda (\lambda > 0)$ denote the unique solution of equation $Ax = \lambda x$. Then

1. x_λ is strictly decreasing in λ : $0 < \lambda_1 < \lambda_2$ yields $x_{\lambda_1} > x_{\lambda_2}$;
2. If there exists $\gamma \in (0,1)$ such that $\phi(t) \geq t^\gamma$ for $t \in (0,1)$, then x_λ is continuous in λ : $\lambda \rightarrow \lambda_0 (\lambda_0 > 0)$ yields $\|x_\lambda - x_{\lambda_0}\| \rightarrow 0$;
3. $\lim_{\lambda \rightarrow \infty} \|x_\lambda\| = 0$, $\lim_{\lambda \rightarrow 0} \|x_\lambda\| = \infty$.

Lemma 2.5 ([43]). Let E, K, h be given as in Lemma 2.3, and suppose that $A: K \rightarrow K$ is a decreasing operator which satisfies:

1. There is $h_0 \in K_h$ such that $Ah_0 \in K_h$;
2. For $x \in K$ and $t \in (0,1)$, there exists $\phi(t) \in (t,1)$ such that $A(tx) \leq \frac{1}{\phi(t)}Ax$.

Then the operator A has a unique fixed point $x^* \in K_h$. And, taking any initial value $x_0 \in K_h$, we make the sequence $x_n = Ax_{n-1}$, $n = 1, 2, \dots$, then $\|x_n - x^*\| \rightarrow 0$ as $n \rightarrow \infty$.

Lemma 2.6. Under the conditions of Lemma 2.5 and let $x_\lambda (\lambda > 0)$ denote the unique solution of equation $Ax = \lambda x$. Then

1. x_λ is strictly decreasing in λ : $0 < \lambda_1 < \lambda_2$ yields $x_{\lambda_1} > x_{\lambda_2}$;
2. If there is $\gamma \in (0,1)$ such that $\frac{1}{\phi(t)} \leq \frac{1}{t^\gamma}$ for $t \in (0,1)$, then x_λ is continuous in λ . Namely, $\lambda \rightarrow \lambda_0 (\lambda_0 > 0)$ yields $\|x_\lambda - x_{\lambda_0}\| \rightarrow 0$;
3. $\lim_{\lambda \rightarrow \infty} \|x_\lambda\| = 0$, $\lim_{\lambda \rightarrow 0} \|x_\lambda\| = \infty$.

Proof. For $\lambda > 0$, by Lemma 2.5, $\frac{1}{\lambda}A: K_h \rightarrow K_h$ is a decreasing operator and satisfies

$$\left(\frac{1}{\lambda}A\right)(tx) = \frac{1}{\lambda}A(tx) \leq \frac{1}{\lambda} \cdot \frac{1}{\phi(t)}Ax = \frac{1}{\phi(t)}\left(\frac{1}{\lambda}A\right)(x), \quad x \in K_h, t \in (0,1).$$

Then, according to Lemma 2.5, $\frac{1}{\lambda}A$ has a unique fixed point x_λ in K_h , that is to say, $Ax_\lambda = \lambda x_\lambda$.

Let $0 < \lambda_1 < \lambda_2$ and define $t_0 = \sup\{t > 0: x_{\lambda_1} \geq tx_{\lambda_2}\}$. Then $0 < t_0 < \infty$ and $x_{\lambda_1} \geq t_0 x_{\lambda_2}$. Next we prove $t_0 \geq 1$. In fact, if $t_0 < 1$, then according to Lemma 2.5 (ii), we can get

$$x_{\lambda_1} = \frac{1}{\lambda_1}Ax_{\lambda_1} \leq \frac{1}{\lambda_1}A(t_0 x_{\lambda_2}) \leq \frac{1}{\lambda_1} \frac{1}{\phi(t_0)}Ax_{\lambda_2} \leq \frac{1}{\lambda_1} t_0^{-\gamma} Ax_{\lambda_2} = \frac{\lambda_2}{\lambda_1} t_0^{-\gamma} x_{\lambda_2},$$

and thus

$$x_{\lambda_2} \geq \frac{\lambda_1}{\lambda_2} t_0^\gamma x_{\lambda_1}.$$

Hence,

$$x_{\lambda_2} = \frac{1}{\lambda_2} Ax_{\lambda_2} \leq \frac{1}{\lambda_2} A\left(\frac{\lambda_1}{\lambda_2} t_0^\gamma x_{\lambda_1}\right) \leq \frac{1}{\lambda_2} [\phi(\frac{\lambda_1}{\lambda_2} t_0^\gamma)]^{-1} Ax_{\lambda_1} \leq \frac{1}{\lambda_2} (\frac{\lambda_1}{\lambda_2} t_0^\gamma)^{-\gamma} Ax_{\lambda_1} = (\frac{\lambda_1}{\lambda_2})^{1-\gamma} t_0^{-\gamma^2} x_{\lambda_1},$$

that is,

$$x_{\lambda_1} \geq (\frac{\lambda_2}{\lambda_1})^{1-\gamma} t_0^{\gamma^2} x_{\lambda_2}.$$

Since $(\frac{\lambda_2}{\lambda_1})^{1-\gamma} t_0^{\gamma^2} > t_0$, this contradicts the definition of t_0 . Thus, $t_0 \geq 1$. Therefore, $x_{\lambda_1} \geq x_{\lambda_2}$. From this, we further know that $x_{\lambda_1} = \frac{1}{\lambda_1} Ax_{\lambda_1} \leq \frac{1}{\lambda_1} Ax_{\lambda_2} = \frac{\lambda_2}{\lambda_1} x_{\lambda_2}$, that means $x_{\lambda_2} \geq \frac{\lambda_1}{\lambda_2} x_{\lambda_1}$. By using of that A is increasing,

$$x_{\lambda_2} = \frac{1}{\lambda_2} Ax_{\lambda_2} \leq \frac{1}{\lambda_2} A\left(\frac{\lambda_1}{\lambda_2} x_{\lambda_1}\right) \leq \frac{1}{\lambda_2} [\phi(\frac{\lambda_1}{\lambda_2})]^{-1} Ax_{\lambda_1} \leq \frac{1}{\lambda_2} (\frac{\lambda_1}{\lambda_2})^{-\gamma} Ax_{\lambda_1} = (\frac{\lambda_1}{\lambda_2})^{1-\gamma} x_{\lambda_1},$$

and hence

$$x_{\lambda_1} \geq (\frac{\lambda_2}{\lambda_1})^{1-\gamma} x_{\lambda_2} \gg x_{\lambda_2}, \quad \forall 0 < \lambda_1 < \lambda_2. \quad (2.1)$$

Now we prove that x_λ is continuous in λ . Given $\lambda_0 > 0$. From (2.1),

$$x_\lambda < x_{\lambda_0}, \quad \forall \lambda > \lambda_0. \quad (2.2)$$

Let $t_\lambda = \sup\{t > 0: x_\lambda \geq tx_{\lambda_0}\}$ ($\lambda > \lambda_0$). Then $0 < t_\lambda < 1$ and $x_\lambda \geq t_\lambda x_{\lambda_0}$. Thus

$$x_\lambda = \frac{1}{\lambda} Ax_\lambda \leq \frac{1}{\lambda} A(t_\lambda x_{\lambda_0}) \leq \frac{1}{\lambda} \frac{1}{\phi(t_\lambda)} Ax_{\lambda_0} \leq \frac{1}{\lambda} t_\lambda^{-\gamma} Ax_{\lambda_0} = \frac{\lambda_0}{\lambda} t_\lambda^{-\gamma} x_{\lambda_0},$$

that is to say, $x_{\lambda_0} \geq \frac{\lambda}{\lambda_0} t_\lambda^\gamma x_\lambda \gg t_\lambda^\gamma x_\lambda$. Consequently,

$$x_{\lambda_0} = \frac{1}{\lambda_0} Ax_{\lambda_0} \leq \frac{1}{\lambda_0} A(t_\lambda^\gamma x_\lambda) \leq \frac{1}{\lambda_0} [\phi(t_\lambda^\gamma x_\lambda)]^{-1} Ax_\lambda \leq \frac{1}{\lambda_0} (t_\lambda^\gamma)^{-\gamma} Ax_\lambda = \frac{\lambda}{\lambda_0} t_\lambda^{-\gamma^2} x_\lambda,$$

which implies

$$x_\lambda \geq \frac{\lambda_0}{\lambda} t_\lambda^{\gamma^2} x_{\lambda_0}.$$

By the definition of t_λ , we obtain $t_\lambda \geq \frac{\lambda_0}{\lambda} t_\lambda^{\gamma^2}$, that is, $t_\lambda^{1-\gamma^2} \geq \frac{\lambda_0}{\lambda}$. Since $1 - \gamma^2 > 0$, it follows that $t_\lambda \geq (\frac{\lambda_0}{\lambda})^{\frac{1}{1-\gamma^2}}$. Therefore,

$$x_\lambda \geq \left(\frac{\lambda_0}{\lambda}\right)^{\frac{1}{1-\gamma^2}} x_{\lambda_0}, \quad \forall \lambda > \lambda_0. \quad (2.3)$$

Let N be the normal constant of K . By (2.2) and (2.3), we obtain

$$\|x_{\lambda_0} - x_\lambda\| \leq \left\| x_{\lambda_0} - \left(\frac{\lambda_0}{\lambda}\right)^{\frac{1}{1-\gamma^2}} x_{\lambda_0} \right\| \leq N \left[1 - \left(\frac{\lambda_0}{\lambda}\right)^{\frac{1}{1-\gamma^2}} \right] \|x_{\lambda_0}\|, \quad \lambda > \lambda_0.$$

From this, we know that when $\lambda \rightarrow \lambda_0 + 0$, $\|x_\lambda - x_{\lambda_0}\| \rightarrow 0$. Similarly, it can be proved that when $\lambda \rightarrow \lambda_0 - 0$, $\|x_\lambda - x_{\lambda_0}\| \rightarrow 0$.

In (2.1), let $\lambda_1 = 1$ and $\lambda_2 = \lambda$, then

$$x_1 \geq \lambda^{1-\gamma} x_\lambda, \quad \forall \lambda > 1,$$

and thus

$$\|x_\lambda\| \leq \frac{N}{\lambda^{1-\gamma}} \|x_1\|, \quad \forall \lambda > 1,$$

therefore, when $\lambda \rightarrow \infty$, $\|x_\lambda\| \rightarrow 0$. In (2.1), take $\lambda_1 = \lambda$ and $\lambda_2 = 1$, then

$$x_\lambda \geq \frac{1}{\lambda^{1-\gamma}} x_1, \quad \forall 0 < \lambda < 1,$$

so,

$$\|x_\lambda\| \geq \frac{1}{N\lambda^{1-\gamma}} \|x_1\|, \quad \forall 0 < \lambda < 1.$$

From this, we know that when $\lambda \rightarrow 0^+$, $\|x_\lambda\| \rightarrow \infty$.

3. Main Results

We discuss the problem (1.1) in the space $E = C[0,1]$, a Banach space under the norm $\|Y\| = \max_{0 \leq t \leq 1} |Y(t)|$, where $Y \in C[0,1]$. Define a cone $K \subset E$ by $K = \{Y \in E: Y(t) \geq 0, t \in [0,1]\}$, then K is normal.

Theorem 3.1. Assume that

(H1) $g(t, Y): [0,1] \times [0, +\infty) \rightarrow [0, +\infty)$ is continuous, $g(t, 0) \not\equiv 0$ for $t \in [0,1]$ and

$$0 < \int_0^1 (1-s)^{\alpha-2} g\left(s, \frac{(\alpha-1)^{\alpha-1}}{\alpha^\alpha}\right) ds < +\infty;$$

(H2) $g(t, Y)$ is increasing in $Y \in [0, +\infty)$ for fixed $t \in [0,1]$;

(H3) For $\tau \in (0,1)$, there exists $\phi: (0,1) \rightarrow (0,1)$ with $\phi(\tau) > \tau$ such that $g(t, \tau Y) \geq \phi(\tau)g(t, Y)$, $Y \in [0, +\infty), t \in [0,1]$.

Then

1. For every fixed $\lambda > 0$, the problem (1.1) has a unique positive solution Y_λ^* in K_h , where $h(t) = t^{\alpha-1} - t^\alpha$ for $t \in [0,1]$. And, for $Y_0 \in K_h$, putting the sequence

$$Y_n(t) = \lambda \int_0^1 t^{\alpha-2} k_\alpha(t, s) g(s, Y_{n-1}(s)) ds, n = 1, 2, \dots,$$

it must be that $Y_n(t) \rightarrow Y_\lambda^*(t)$ as $n \rightarrow +\infty$, where $k_\alpha(t, s)$ is given as in Lemma 2.1;

2. Y_λ^* is strictly increasing in λ : $0 < \lambda_1 < \lambda_2$ means $Y_{\lambda_1}^* < Y_{\lambda_2}^*$;

3. If there is $\gamma \in (0,1)$ such that $\phi(\tau) \geq \tau^\gamma$ ($0 < \tau < 1$), then Y_λ^* is continuous in λ : $\lambda \rightarrow \lambda_0$ ($\lambda_0 > 0$) means $\|Y_\lambda^* - Y_{\lambda_0}^*\| \rightarrow 0$;

4.. $\lim_{\lambda \rightarrow 0} \|Y_\lambda^*\| = 0$, $\lim_{\lambda \rightarrow +\infty} \|Y_\lambda^*\| = +\infty$.

Proof. Now, define an operator $T: E \rightarrow E$ by

$$TY(t) = \int_0^1 t^{\alpha-2} k_\alpha(t, s) g(s, Y(s)) ds,$$

where $k_\alpha(t, s)$ is given by Lemma 2.1. Firstly, we prove that $T: K \rightarrow K$. For arbitrary $Y \in K$, we have $Y(t) \geq 0$ for $t \in [0, 1]$. By Lemma 2.2 and (H_1) ,

$$TY(t) = \int_0^1 t^{\alpha-2} k_\alpha(t, s) g(s, Y(s)) ds \geq \int_0^1 t^{\alpha-2} \frac{(1-\Lambda)p(t)q(s)}{\Gamma(\alpha-1)} g(s, Y(s)) ds \geq 0,$$

and according to the definition of cone K , $T: K \rightarrow K$. Next, we verify that $T: K \rightarrow K$ is increasing. For any $Y_1, Y_2 \in K$ and suppose $Y_1 \leq Y_2$, we get $Y_1(t) \leq Y_2(t)$, $t \in [0, 1]$. So for $t \in [0, 1]$, by Lemma 2.2 and (H_2) ,

$$\begin{aligned} TY_2(t) - TY_1(t) &= \int_0^1 t^{\alpha-2} k_\alpha(t, s) [g(s, Y_2(s)) - g(s, Y_1(s))] ds \\ &\geq \int_0^1 t^{\alpha-2} \frac{(1-\Lambda)p(t)q(s)}{\Gamma(\alpha-1)} [g(s, Y_2(s)) - g(s, Y_1(s))] ds \geq 0, \end{aligned}$$

this shows that $T: K \rightarrow K$ is increasing. In the sequel, we check that T satisfies other conditions of Lemma 2.3. Since $h(t) = t^{\alpha-2}p(t) = t^{\alpha-1} - t^\alpha$, it is obvious to see that $h \in K$. Then we take the derivative of $h(t)$, and there exist

$$\min_{t \in [0, 1]} h(t) = h(0), \max_{t \in [0, 1]} h(t) = h\left(\frac{\alpha-1}{\alpha}\right).$$

Next, we mainly show that $Th \in K_h$. Let $t_0 = \frac{\alpha-1}{\alpha}$, according to the continuity of $g(t, Y)$ on $[0, 1] \times [0, +\infty)$ and (H_2) , we can obtain $g(t, h(t)) \leq g(t, h(t_0))$ for $t \in [0, 1]$. Further, considering Lemma 2.2, the following inequalities hold

$$\begin{aligned} Th(t) &= \int_0^1 t^{\alpha-2} k_\alpha(t, s) g(s, h(s)) ds \\ &\geq \int_0^1 t^{\alpha-2} \frac{(1-\Lambda)p(t)q(s)}{\Gamma(\alpha-1)} g(s, h(0)) ds \\ &= \frac{(1-\Lambda)}{\Gamma(\alpha-1)} \int_0^1 t^{\alpha-2} p(t) q(s) g(s, 0) ds \\ &= \frac{(1-\Lambda)}{\Gamma(\alpha-1)} \int_0^1 q(s) g(s, 0) ds \cdot h(t) \\ &:= l_1 h(t), \end{aligned}$$

$$\begin{aligned} Th(t) &= \int_0^1 t^{\alpha-2} k_\alpha(t, s) g(s, h(s)) ds \\ &\leq \int_0^1 t^{\alpha-2} \frac{(1+\Lambda)p(t)q(s)}{\Gamma(\alpha)p(s)} g(s, h(t_0)) ds \\ &= \frac{(1+\Lambda)}{\Gamma(\alpha)} \int_0^1 \frac{q(s)}{p(s)} g(s, h(t_0)) ds \cdot h(t) \\ &= \frac{(1+\Lambda)}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-2} g\left(s, \frac{(\alpha-1)^{\alpha-1}}{\alpha^\alpha}\right) ds \cdot h(t) \\ &:= l_2 h(t), \end{aligned}$$

where $l_1 = \frac{(1-\Lambda)}{\Gamma(\alpha-1)} \int_0^1 q(s) g(s, 0) ds$, $l_2 = \frac{(1+\Lambda)}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-2} g\left(s, \frac{(\alpha-1)^{\alpha-1}}{\alpha^\alpha}\right) ds$. From (H_1) and Lemma 2.2, we know that $q(s)g(s, 0) \geq 0$, $q(s)g(s, 0) \not\equiv 0$ for $s \in [0, 1]$ and $\Lambda < 1$, so $l_1 > 0$. Also, by Lemma 2.2 (i)(ii) and (H_1) , we can easily get $l_2 \geq l_1$. This further implies that, $0 < l_1 \leq l_2$. That means, $l_1 h \leq Th \leq l_2 h$, so $Th \in K_h$.

Now, we indicate that the condition (2) of Lemma 2.4 is also satisfied. For $\tau \in (0, 1)$, $Y \in K$, by (H_3) ,

$$\begin{aligned}
T(\tau Y)(t) &= \int_0^1 t^{\alpha-2} k_\alpha(t, s) g(s, \tau Y(s)) ds \\
&\geq \phi(\tau) \int_0^1 t^{\alpha-2} k_\alpha(t, s) g(s, Y(s)) ds \\
&= \phi(\tau) TY(t).
\end{aligned}$$

Considering $Y(t) = \lambda TY(t)$. From the above content, it can be seen that T satisfies the conditions of Lemma 2.3. Consequently, there exists a unique $Y_\lambda^* \in K_h$ such that $TY_\lambda^* = \frac{1}{\lambda} Y_\lambda^*$, that is $Y_\lambda^* = \lambda TY_\lambda^*$ and

$$Y_\lambda^*(t) = \lambda \int_0^1 t^{\alpha-2} k_\alpha(t, s) g(s, Y_\lambda^*(s)) ds.$$

Let $\mu = \frac{1}{\lambda}$, then $TY_\lambda^* = \mu Y_\lambda^*$. From Lemma 2.1, $Y_\lambda^* = Y_\mu^*$ is a unique positive solution of (1.1). Further by Lemma 2.4 (1), Y_μ^* is strictly decreasing in μ . That is, Y_λ^* is strictly increasing in λ : $Y_{\lambda_1}^* < Y_{\lambda_2}^*$ for $0 < \lambda_1 < \lambda_2$; by Lemma 2.4 (3), one has $\lim_{\lambda \rightarrow +\infty} \|Y_\lambda^*\| = +\infty$, $\lim_{\lambda \rightarrow 0} \|Y_\lambda^*\| = 0$. Moreover, if $\phi(\tau) \geq \tau^\gamma$ for $\tau \in (0, 1)$, then Lemma 2.4 (2) implies that Y_λ^* is continuous in λ : $\lambda \rightarrow \lambda_0 (\lambda_0 > 0)$ implies $\|Y_\lambda^* - Y_{\lambda_0}^*\| \rightarrow 0$. In addition, let $\tilde{T} = \lambda T$, then $\tilde{T}$ also satisfies all the conditions of Lemma 2.3, and thus, for $Y_0 \in K_h$, putting the sequence $Y_n = \tilde{T}Y_{n-1}$, $n = 1, 2, \dots$, one has $Y_n \rightarrow Y_\lambda^*$ as $n \rightarrow +\infty$. In other words,

$$Y_n(t) = \lambda \int_0^1 t^{\alpha-2} k_\alpha(t, s) g(s, Y_{n-1}(s)) ds, n = 1, 2, \dots$$

and $Y_n(t) \rightarrow Y_\lambda^*(t)$ as $n \rightarrow +\infty$. Noting that $Y_\lambda^* \in K_h$ and $h(0) = h(1) = 0$, $h(t) > 0$ for $t \in (0, 1)$, we get $Y_\lambda^*(t) > 0$ for $t \in (0, 1)$. So Y_λ^* is a positive solution.

Corollary 3.1. Assume that conditions (H1)(H2)(H3) in Theorem 3.1 are true, we take $\lambda = 1$, it can be known that the FDE with Dirichlet BCs

$$\begin{cases} -D_{0+}^\alpha Y(t) + a(t)Y(t) = g(t, Y(t)), & 0 < t < 1, \alpha \in (1, 2), \\ Y(0) = 0, \quad Y(1) = 0, \end{cases}$$

has a unique positive solution Y^* in K_h , where $h(t) = t^{\alpha-1} - t^\alpha$. And, for $Y_0 \in K_h$, making the sequence

$$Y_n(t) = \int_0^1 t^{\alpha-2} k_\alpha(t, s) g(s, Y_{n-1}(s)) ds, n = 1, 2, \dots,$$

one has $Y_n(t) \rightarrow Y^*(t)$ as $n \rightarrow +\infty$, where $a \in L^1[0, 1]$, $k_\alpha(t, s) = \sum_{n=0}^{+\infty} (-1)^n k_n(t, s)$, and $k_n(t, s) = \int_0^1 a(\eta) \eta^{\alpha-2} k_0(t, \eta) k_{n-1}(\eta, s) d\eta$ ($n = 1, 2, \dots$).

Theorem 3.2. Suppose that

(H4) $g(t, Y): [0, 1] \times [0, +\infty) \rightarrow [0, +\infty)$ is continuous, decreasing in $Y \in [0, +\infty)$ for fixed $t \in [0, 1]$, $g(t, h(t_0)) \not\equiv 0$, $t \in [0, 1]$ and

$$0 < \int_0^1 (1-s)^{\alpha-2} g(s, 0) ds < +\infty;$$

(H5) For $\mu \in (0, 1)$, there is $\phi(\mu) \in (\mu, 1)$ such that $g(t, \mu Y) \leq \frac{1}{\phi(\mu)} g(t, Y)$, $t \in [0, 1]$, $Y \in [0, +\infty)$.

Then conclusions 1, 2, 4 in Theorem 3.1 hold. Further, if there is $\gamma \in (0, 1)$ such that $\frac{1}{\phi(\mu)} \leq \frac{1}{\mu^\gamma}$ for $\mu \in (0, 1)$, then Y_λ^* is continuous in λ : $\lambda \rightarrow \lambda_0 (\lambda_0 > 0)$ yields $\|Y_\lambda^* - Y_{\lambda_0}^*\| \rightarrow 0$.

Proof. From Theorem 3.1, $T: K \rightarrow K$. Next, we can prove that $T: K \rightarrow K$ is decreasing. For any $Y_1, Y_2 \in K$ and let $Y_1 \leq Y_2$, then we get $Y_1(t) \leq Y_2(t), t \in [0, 1]$. So for any $t \in [0, 1]$, according to Lemma 2.1 and (H4),

$$\begin{aligned} TY_2(t) - TY_1(t) &= \int_0^1 t^{\alpha-2} k_\alpha(t, s) [g(s, Y_2(s)) - g(s, Y_1(s))] ds \\ &\leq \int_0^1 t^{\alpha-2} \frac{(1+\Lambda)q(s)}{\Gamma(\alpha)} [g(s, Y_2(s)) - g(s, Y_1(s))] ds \leq 0. \end{aligned}$$

This implies that $T: K \rightarrow K$ is a decreasing operator. Meanwhile, according to Lemma 2.2,

$$\begin{aligned} Th(t) &= \int_0^1 t^{\alpha-2} k_\alpha(t, s) g(s, h(s)) ds \\ &\geq \int_0^1 t^{\alpha-2} \frac{(1-\Lambda)p(t)q(s)}{\Gamma(\alpha-1)} g(s, h(t_0)) ds \\ &= \frac{(1-\Lambda)}{\Gamma(\alpha-1)} \int_0^1 q(s) g(s, h(t_0)) ds \cdot h(t) \\ &:= l_3 h(t), \end{aligned}$$

$$\begin{aligned} Th(t) &= \int_0^1 t^{\alpha-2} k_\alpha(t, s) g(s, h(s)) ds \\ &\leq \int_0^1 t^{\alpha-2} \frac{(1+\Lambda)p(t)q(s)}{\Gamma(\alpha)p(s)} g(s, h(0)) ds \\ &= \frac{(1+\Lambda)}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-2} g(s, 0) ds \cdot h(t) \\ &:= l_4 h(t), \end{aligned}$$

where $l_3 = \frac{(1-\Lambda)}{\Gamma(\alpha-1)} \int_0^1 q(s) g(s, h(t_0)) ds$, $l_4 = \frac{(1+\Lambda)}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-2} g(s, 0) ds$. From (H4) and Lemma 2.2, we get that $q(s)g(s, h(t_0)) \geq 0, q(s)g(s, h(t_0)) \not\equiv 0, \Lambda < 1$ and $\frac{1-\Lambda}{\Gamma(\alpha-1)} \leq \frac{1+\Lambda}{\Gamma(\alpha)p(s)}$, so $0 < l_3$. Also, by Lemma 2.2 (i) (ii) and (H4), we can obtain $l_3 \leq l_4$ and thus $l_3 h \leq Th \leq l_4 h$, so $Th \in K_h$.

Additionally, from (H5), the following inequality holds:

$$\begin{aligned} T(\mu Y)(t) &= \int_0^1 t^{\alpha-2} k_\alpha(t, s) g(s, \mu Y(s)) ds \\ &\leq \frac{1}{\phi(\mu)} \int_0^1 t^{\alpha-2} k_\alpha(t, s) g(s, Y(s)) ds \\ &= \frac{1}{\phi(\mu)} TY(t). \end{aligned}$$

Similarly, we consider $Y(t) = \lambda TY(t)$. From the above content, it can be seen that T satisfies the conditions of Lemma 2.5. Consequently, there exists a unique $Y_\lambda^* \in K_h$ such that $TY_\lambda^* = \frac{1}{\lambda} Y_\lambda^*$, that is $Y_\lambda^* = \lambda TY_\lambda^*$ and

$$Y_\lambda^*(t) = \lambda \int_0^1 t^{\alpha-2} k_\alpha(t, s) g(s, Y_\lambda^*(s)) ds.$$

From Lemma 2.1, Y_λ^* is a unique positive solution of (1.1). Further, by Lemma 2.6 (1), Y_λ^* is strictly increasing in λ , that is, $Y_{\lambda_1}^* < Y_{\lambda_2}^*$ for $0 < \lambda_1 < \lambda_2$; by Lemma 2.6 (3), we have $\lim_{\lambda \rightarrow +\infty} \|Y_\lambda^*\| = +\infty, \lim_{\lambda \rightarrow 0} \|Y_\lambda^*\| = 0$. Moreover, if $\frac{1}{\phi(\mu)} \leq \frac{1}{\mu^\gamma}$ for $\mu \in (0, 1)$, then Lemma 2.6 (2) yields that Y_λ^* is continuous in λ : $\lambda \rightarrow \lambda_0 (\lambda_0 > 0)$ implies $\|Y_\lambda^* - Y_{\lambda_0}^*\| \rightarrow 0$. Further, let $\tilde{T} = \lambda T$, then $\tilde{T}$ also satisfies all the conditions of Lemma 2.5, and thus, for $Y_0 \in K_h$, putting the sequence $Y_n = \tilde{T}Y_{n-1}, n = 1, 2, \dots$, we have $Y_n \rightarrow Y_\lambda^*$ as $n \rightarrow +\infty$. In other words,

$$Y_n(t) = \lambda \int_0^1 t^{\alpha-2} k_\alpha(t, s) g(s, Y_{n-1}(s)) ds, n = 1, 2, \dots,$$

and $Y_n(t) \rightarrow Y_\lambda^*(t)$ as $n \rightarrow +\infty$. W

Corollary 3.2 Assume that conditions (H1)(H4)(H5) in Theorem 3.2 are true, we take $\lambda = 1$, it can be known that the FDEn with Dirichlet BCs

$$\begin{cases} -D_{0+}^\alpha Y(t) + a(t)Y(t) = g(t, Y(t)), & 0 < t < 1, \alpha \in (1, 2), \\ Y(0) = 0, & Y(1) = 0 \end{cases}$$

has a unique positive solution Y^* in K_h , where $h(t) = t^{\alpha-1} - t^\alpha$. Further, for $Y_0 \in K_h$, making the sequence

$$Y_n(t) = \int_0^1 t^{\alpha-2} k_\alpha(t, s) g(s, Y_{n-1}(s)) ds, n = 1, 2, \dots,$$

we get $Y_n(t) \rightarrow Y^*(t)$ as $n \rightarrow +\infty$, where $a \in L^1[0, 1]$, $k_\alpha(t, s) = \sum_{n=0}^{+\infty} (-1)^n k_n(t, s)$, and $k_n(t, s) = \int_0^1 a(\eta) \eta^{\alpha-2} k_0(t, \eta) k_{n-1}(\eta, s) d\eta, (n = 1, 2, \dots)$.

4. Examples

Example 4.1. Consider the following FBVP with a positive parameter

$$\begin{cases} -D_{0+}^{\frac{3}{2}} Y(t) - \frac{1}{40} t^{\frac{1}{2}} (1-t)^{\frac{1}{2}} Y(t) = \lambda [\frac{1}{2} + (Y(t))^{\frac{1}{6}}] (1-t)^{\frac{1}{2}}, & t \in [0, 1], \\ Y(0) = 0, & Y(1) = 0. \end{cases} \quad (4.1)$$

In this problem, $\alpha = \frac{3}{2}$, $a(t) = -\frac{1}{40} t^{\frac{1}{2}} (1-t)^{\frac{1}{2}}$, $g(t, Y) = [\frac{1}{2} + Y^{\frac{1}{6}}] (1-t)^{\frac{1}{2}}$ and

$$\Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2} \approx 0.88623,$$

$$\Lambda_1 = \frac{1}{40} \int_0^1 t^{\frac{1}{2}} (1-t)^{\frac{1}{2}} [t(1-t)]^{\frac{1}{2}} dt = \frac{1}{240} < \Gamma\left(\frac{3}{2}\right),$$

$$\Lambda_2 = \frac{1}{40} \int_0^1 t^{\frac{1}{2}} (1-t)^{\frac{1}{2}} [t(1-t)]^{-\frac{1}{2}} dt = \frac{1}{40},$$

$$\Lambda = \frac{\Lambda_2}{\frac{1}{2}(\Gamma(\frac{3}{2}) - \Lambda_1)} \approx 0.05669,$$

$$\int_0^1 (1-s)^{\alpha-2} g\left(s, \frac{(\alpha-1)^{\alpha-1}}{\alpha^\alpha}\right) ds = \int_0^1 (1-s)^{-\frac{1}{2}} (1-s)^{\frac{1}{2}} \left(\frac{1}{2} + \frac{(\frac{1}{2})^{\frac{1}{2}}}{(\frac{3}{2})^{\frac{3}{2}}}\right) ds = \frac{1}{2} + \frac{(\frac{1}{2})^{\frac{1}{2}}}{(\frac{3}{2})^{\frac{3}{2}}}.$$

Evidently, $a \in L^1[0, 1]$ and $g(t, Y)$ is a continuous function and $g(t, 0) \neq 0$ for $t \in [0, 1]$. $g(t, Y)$ is an increasing function in Y . Then, it satisfies conditions (H_1) , (H_2) .

Let $\phi(\tau) = \tau^{\frac{1}{6}}$, $\gamma = \frac{1}{4}$, then $\phi(\tau) \in (\tau, 1)$ for $\tau \in (0, 1)$, and $\phi(\tau) > \tau^\gamma$. For $Y \in [0, +\infty)$,

$$g(t, \tau Y) = \left((\tau Y)^{\frac{1}{6}} + \frac{1}{2}\right) (1-t)^{\frac{1}{2}} \geq \tau^{\frac{1}{6}} (Y^{\frac{1}{6}} + \frac{1}{2}) (1-t)^{\frac{1}{2}} = \phi(\tau) g(t, Y),$$

which satisfies (H_3) . From Theorem 3.1, we can claim that

1. For $\lambda > 0$, (4.1) has a unique positive solution Y_λ^* in K_h , $t \in [0,1]$. Moreover, for $Y_0 \in K_h$, the sequence

$$Y_n(t) = \lambda \int_0^1 t^{\alpha-2} k_\alpha(t, s) \left[\frac{1}{2} + (Y_{n-1}(s))^{\frac{1}{6}} \right] (1-s)^{\frac{1}{2}} ds, \quad n = 1, 2, \dots,$$

satisfies that $Y_n(t) \rightarrow Y_\lambda^*(t)$ as $n \rightarrow \infty$, where $k_\alpha(t, s)$ is given in Lemma 2.1;

2. Y_λ^* is strictly increasing in λ : $Y_{\lambda_1}^* < Y_{\lambda_2}^*$ for $0 < \lambda_1 < \lambda_2$;

3. Y_λ^* is continuous in λ : $\|Y_\lambda^* - Y_{\lambda_0}^*\| \rightarrow 0$ as $\lambda \rightarrow \lambda_0$ ($\lambda_0 > 0$);

4. $\lim_{\lambda \rightarrow 0} \|u_\lambda^*\| = 0$, $\lim_{\lambda \rightarrow +\infty} \|u_\lambda^*\| = +\infty$.

Example 4.2. For the following FBVP

$$\begin{cases} -D_{0+}^{\frac{3}{2}} Y(t) - \frac{1}{40} t^{\frac{1}{2}} (1-t)^{\frac{1}{2}} Y(t) = \lambda \frac{(1-t)^{\frac{1}{2}}}{\left[1+(Y(t))^{\frac{1}{3}}\right]}, & t \in [0,1], \\ Y(0) = 0, \quad Y(1) = 0, \end{cases} \quad (4.2)$$

where $g(t, Y) = \frac{(1-t)^{\frac{1}{2}}}{\left[1+Y^{\frac{1}{3}}\right]}$ and $\Lambda_1, \Lambda_2, \Lambda$ are the same as in Example 4.1. Moreover,

$$\int_0^1 (1-s)^{\alpha-2} g(s, 0) ds = \int_0^1 (1-s)^{-\frac{1}{2}} g(s, 0) ds = 1.$$

Clearly, $g(t, Y)$ is a continuous function and $g(t, h(t_0)) \neq 0$ for $t \in [0,1]$, where $t_0 = \frac{1}{3}$. Obviously, $g(t, Y)$ is a decreasing function with respect to $Y \geq 0$. So it satisfies condition (H_4) .

Let $\phi(\mu) = \mu^{\frac{1}{3}}$, $\gamma = \frac{2}{3}$, then $\phi(\mu) \in (\mu, 1)$ for $\mu \in (0,1)$ and $\frac{1}{\phi(\mu)} \leq \frac{1}{\mu^\gamma}$. For $Y \in [0, \infty)$,

$$g(t, \mu Y) = \frac{(1-t)^{\frac{1}{2}}}{\left[(\mu Y)^{\frac{1}{3}} + 1\right]} \leq \frac{(1-t)^{\frac{1}{2}}}{\mu^{\frac{1}{3}}(Y^{\frac{1}{3}} + 1)} = \frac{1}{\phi(\mu)} g(t, Y),$$

which satisfies (H_5) . Then, according to Theorem 3.2, for $\lambda > 0$, (4.2) has a unique positive solution Y_λ^* in K_h . Moreover, for $Y_0 \in K_h$, the sequence

$$Y_n(t) = \lambda \int_0^1 t^{\alpha-2} k_\alpha(t, s) \frac{(1-s)^{\frac{1}{2}}}{\left[1+(Y_{n-1}(s))^{\frac{1}{3}}\right]} ds, \quad n = 1, 2, \dots,$$

satisfies that $Y_n(t) \rightarrow Y_\lambda^*(t)$ as $n \rightarrow \infty$, where $k_\alpha(t, s)$ is given in Lemma 2.1. At the same time, the conclusions (b)(c)(d) in Example 4.1 are also true.

Conflict of Interest

The authors declare that they have no competing interests.

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Authors' Contributions

The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript. Chengbo Zhai: Actualization, methodology, formal analysis, validation and was a major contributor in writing the manuscript. Xiaoyan Liu: investigation and initial draft. Yong Zhou: Actualization, methodology, formal analysis, validation, writing. Chen Yang: Actualization, methodology, formal analysis, validation.

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