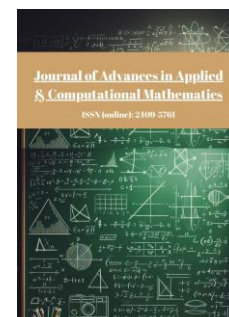




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A Memory-dependent (q, τ) -deformed Fractional Decision-support Model for Sustainable Religious Tourism Management in Iraq

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ABSTRACT

This study proposes a memory-dependent (q, τ) -deformed fractional decision model for religious tourism management in Iraq, with a case study focused on the Najaf-Karbala pilgrimage corridor during high-density visitation periods such as Arbaeen. The model describes the nonlinear interaction among pilgrim flow, crowding intensity, service capacity, risk pressure, and visitor satisfaction. The deformation parameters (q, τ) capture scale-sensitive and delayed responses in tourism demand, whereas the fractional order α represents memory effects associated with seasonal pilgrimage behavior and accumulated service pressure. The framework provides a quantitative tool for forecasting congestion, evaluating service-capacity limitations, and supporting sustainable planning for religious-tourism destinations in Iraq.

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1. Introduction

Time-fractional evolution equations provide a powerful mathematical framework for describing dynamical processes whose present behavior depends on their past history [1-3]. Unlike classical evolution equations, which are governed by local integer-order derivatives, time-fractional models involve nonlocal temporal operators and therefore naturally represent memory, hereditary effects, anomalous relaxation, and delayed responses [4-6]. Such features are particularly relevant to the present tourism-management setting, where the current pilgrim flow, congestion, service pressure, risk, and visitor satisfaction may depend on conditions accumulated over preceding days or previous visitation periods [7-9]. In the proposed framework, the fractional order $\alpha \in (0,1]$ controls the strength of this temporal memory: values of α close to 1 correspond to near-classical short-memory dynamics, whereas smaller values of α produce a stronger dependence on historical states [10-12]. Thus, the proposed system may be interpreted as a nonlinear time-fractional evolution equation whose state vector evolves under a memory-dependent operator [13, 14], providing a more flexible description of delayed and cumulative pilgrimage dynamics than the corresponding classical integer-order model [15, 16].

Religious tourism is one of the most important forms of cultural and spiritual mobility in the contemporary tourism sector. Unlike conventional leisure tourism, religious tourism is driven by faith, collective memory, ritual obligation, sacred geography, and repeated seasonal participation [17, 18]. It is therefore not only an economic activity, but also a social, cultural, and spiritual phenomenon that connects visitors, host communities, religious institutions, public authorities, and service providers. In countries with major pilgrimage destinations, religious tourism creates opportunities for local development, employment, hospitality services, transportation expansion, heritage preservation, and international visibility [19]. At the same time, it generates complex management challenges because visitor flows are often concentrated in short periods, around fixed religious events, and within limited urban and sacred spaces [20].

Iraq provides a highly significant context for studying religious tourism management. The cities of Najaf and Karbala are among the most important religious destinations in the Islamic world, and the Najaf–Karbala corridor forms a major pilgrimage route during the Arbaeen visitation. Arbaeen marks the fortieth day after Ashura and is associated with the commemoration of Imam Hussein. During this period, very large numbers of pilgrims travel to Karbala, many of them walking long distances from Najaf and other regions. The event is not only a pilgrimage movement, but also a large-scale social practice involving hospitality, volunteering, food provision, accommodation, transportation support, medical assistance, and community mobilization. UNESCO inscribed the provision of services and hospitality during the Arbaeen visitation on the Representative List of the Intangible Cultural Heritage of Humanity, emphasizing its importance as a social practice rooted in Iraqi hospitality and collective service [21, 22].

The scale of Arbaeen tourism creates a unique management problem. Millions of visitors may converge toward Karbala within a limited time window, causing intense pressure on roads, public transportation, hotels, temporary shelters, food and water distribution points, sanitation facilities, medical services, and security systems [23, 24]. The pilgrimage route is also affected by spatial concentration, temporal peaks, weather conditions, long walking distances, fatigue, traffic interruption, and uncertainty in visitor arrivals [25]. These conditions make religious tourism management during Arbaeen different from ordinary destination management. The main objective is not only to attract tourists, but to organize mass movement, reduce congestion, protect pilgrims, maintain service quality, preserve the spiritual character of the event, and support sustainable local development [26].

Several practical problems arise during high-density religious tourism periods. First, pilgrim flow is nonlinear. A small increase in visitors may be manageable when service capacity is sufficient, but the same increase can produce severe crowding when the destination approaches its carrying capacity. Second, crowding is cumulative. Congestion in one part of the route can propagate to transport stations, shrine entrances, accommodation areas, medical-service points, and food-distribution locations [27-29]. Third, service capacity is dynamic rather than fixed. Volunteers, buses, ambulances, temporary clinics, sanitation points, and accommodation units may increase before

the event, become overloaded during the peak, and decline after the season. Fourth, risk pressure is multidimensional. It may include health pressure, heat exposure, transport delay, security concerns, emergency-service load, and crowd-safety challenges. Fifth, visitor satisfaction is influenced by both spiritual motivation and management quality [30-32]. Although pilgrims may accept hardship as part of the religious experience, excessive congestion, lack of services, health risks, and poor organization can reduce the overall destination utility and weaken the sustainability of the pilgrimage system.

Traditional tourism-management approaches are often insufficient for this type of problem because they usually treat visitor demand, service capacity, crowding, risk, and satisfaction as separate indicators [33, 34]. However, in Arbaeen tourism these components interact strongly. Increasing pilgrim flow may raise crowding; increasing crowding may increase risk; increasing risk may reduce satisfaction; and increasing service capacity may reduce both crowding and risk [35]. Moreover, these effects are not instantaneous. Previous congestion, past service experience, accumulated fatigue, previous-season reputation, and delayed institutional response can influence the current state of the system. Therefore, a suitable management model should be nonlinear, memory-dependent, scale-sensitive, and capable of representing delayed interactions among the main tourism-management variables.

To address these limitations, this study proposes a (q, τ) -deformed fractional management framework for Arbaeen religious tourism in Iraq [36-38]. The model describes the dynamic interaction among five core variables: pilgrim flow, crowding intensity, service capacity, risk pressure, and visitor satisfaction [39, 40]. The fractional order α is introduced to represent memory effects, meaning that the present tourism state depends not only on current conditions but also on accumulated past conditions. This is important for Arbaeen because pilgrim movement, service pressure, crowding, and satisfaction are influenced by previous days of the event, previous service performance, and repeated seasonal behavior. The deformation parameter q controls the strength of nonlinear deformation in the system response, while the scale parameter τ captures sensitivity to changes in the magnitude of visitor flow and service pressure.

The proposed (q, τ) -management model offers a decision-support structure for religious tourism planning. It allows managers to simulate how different interventions affect the system before, during, and after the pilgrimage peak. For example, increasing transportation capacity, expanding temporary medical units, improving route distribution, increasing water and food stations, or strengthening emergency response can be represented as improvements in service capacity. The model then evaluates how these interventions reduce crowding, lower risk pressure, and improve visitor satisfaction. In this sense, the model transforms religious tourism management from a descriptive process into a predictive and preventive framework. The main contribution of this study is the development of a mathematical tourism-management model that is specifically adapted to the structure of Arbaeen religious tourism in Iraq. Unlike classical models, the proposed framework incorporates memory, nonlinear scale effects, and management intervention mechanisms. It provides a flexible tool for understanding how large pilgrim flows interact with service capacity, crowding, risk, and satisfaction. The study therefore contributes to the growing literature on religious tourism, sustainable destination management, and quantitative decision-support systems for mass pilgrimage events (Fig. 1).

2. Framework for Religious Tourism Management in Iraq

This section develops a nonlinear memory-dependent (q, τ) -deformed fractional model for religious tourism management in Iraq. The proposed framework is motivated by the fact that pilgrimage tourism, especially along the Najaf-Karbala religious corridor, is not a simple short-term visitor arrival process. Instead, it is a highly dynamic system influenced by seasonality, religious-event timing, crowding accumulation, transportation capacity, accommodation availability, public services, health and safety pressure, and the perceived spiritual and service experience of pilgrims. Therefore, the model is designed to describe not only the number of pilgrims arriving at the destination, but also the interaction between demand pressure, service capacity, operational risk, and visitor satisfaction (Table 1).

A Memory-Dependent (q, τ) -Deformed Fractional Decision-Support Model for Sustainable Religious Tourism Management in Iraq

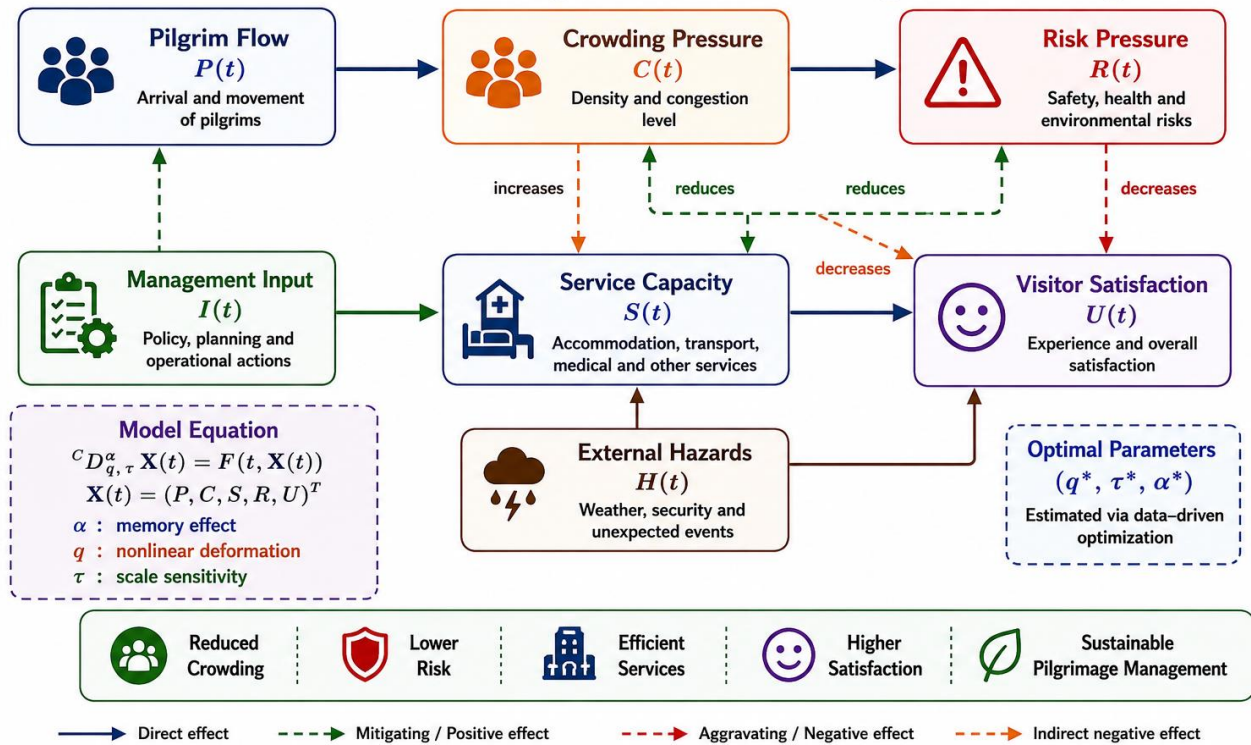


Figure 1: Graphical abstract of the proposed (q, τ) -deformed fractional management model for Arbaeen religious tourism in Iraq. The framework links pilgrim flow, crowding, service capacity, risk pressure, and visitor satisfaction to support safer, smarter, and more sustainable pilgrimage management.

Table 1: Summary of mathematical symbols and notation used in the study.

Symbol	Definition/Interpretation	Domain
State Variables		
t	Time during the religious-tourism or pilgrimage period	$t \geq 0$
$P(t)$	Pilgrim flow or visitor density	$P(t) \geq 0$
$C(t)$	Crowding intensity or congestion pressure	$C(t) \geq 0$
$S(t)$	Available or effective service capacity	$S(t) \geq 0$
$R(t)$	Accumulated risk pressure	$R(t) \geq 0$
$U(t)$	Visitor satisfaction or destination utility	$U(t) \geq 0$
$X(t)$	State vector, $X(t) = (P(t), C(t), S(t), R(t), U(t))^T$	$X(t) \in \mathbb{R}_+^5$
External Inputs		
$I(t)$	Management investment, preparation, or institutional-support input	$I(t) \geq 0$
$H(t)$	External hazard pressure, including weather, heat, transport, health, or security-related effects	$H(t) \geq 0$
Model Parameters		
r	Intrinsic growth rate of pilgrim flow	$r > 0$
K	Effective carrying capacity of the destination system	$K > 0$
a	Strength of the adverse effect of crowding on pilgrim flow	$a > 0$
b	Positive contribution of service capacity to pilgrim flow	$b > 0$

Table 1 (contd....)

Symbol	Definition/Interpretation	Domain
Model Parameters		
λ	Rate at which pilgrim flow generates crowding	$\lambda > 0$
μ	Efficiency of service capacity in reducing crowding	$\mu > 0$
δ	Natural dissipation or relaxation rate of crowding	$\delta > 0$
θ	Efficiency of management investment or preparation in increasing service capacity	$\theta > 0$
γ	Rate of service-capacity consumption caused by pilgrim demand	$\gamma > 0$
ω	Natural depreciation or decay rate of service capacity	$\omega > 0$
ξ	Rate at which crowding increases risk pressure	$\xi > 0$
χ	Sensitivity of risk pressure to external hazards	$\chi > 0$
ψ	Efficiency of service capacity in reducing risk pressure	$\psi > 0$
ν	Natural relaxation or decay rate of risk pressure	$\nu > 0$
β	Increase in visitor satisfaction produced by service capacity	$\beta > 0$
κ	Decrease in visitor satisfaction caused by crowding	$\kappa > 0$
ζ	Decrease in visitor satisfaction caused by risk pressure	$\zeta > 0$
σ	Natural relaxation or decay rate of visitor satisfaction	$\sigma > 0$
Fractional and Deformation Parameters		
α	Fractional order controlling the strength of temporal memory	$0 < \alpha \leq 1$
q	Deformation-intensity parameter	$0 < q \leq 1$
τ	Scale-deformation or scale-sensitivity parameter	$\tau \geq 0$
$CD_{q,\tau}^\alpha$	Caputo-type (q, τ) -deformed fractional derivative of order α	—
$D_{q,\tau}$	Associated (q, τ) -deformed first-order operator	—
$I_{q,\tau}^\alpha$	(q, τ) -deformed fractional integral of order α	—
$\Gamma_{q,\tau}(\alpha)$	(q, τ) -deformed Gamma function	$\alpha > 0$
$(t-s)_{q,\tau}^{\alpha-1}$	(q, τ) -deformed fractional memory kernel appearing in the Volterra integral representation	$0 \leq s < t$
$d_{q,\tau}s$	(q, τ) -deformed integration measure	—
s	Integration or memory-history variable	$0 \leq s \leq t$
Initial Conditions and Analytical Notation		
P_0	Initial pilgrim flow	$P_0 \geq 0$
C_0	Initial crowding level	$C_0 \geq 0$
S_0	Initial service capacity	$S_0 \geq 0$
R_0	Initial risk pressure	$R_0 \geq 0$
U_0	Initial visitor satisfaction or destination utility	$U_0 \geq 0$
X_0	Initial state vector, $X_0 = (P_0, C_0, S_0, R_0, U_0)^T$	R_+^5
$F(t, X)$	Nonlinear vector field of the proposed dynamical system	R^5
$F_i, i = 1, \dots, 5$	Individual components of the vector field F	—
T	Final time of the considered observation interval	$T > 0$
T_0	Local existence time used in the fixed-point argument	$0 < T_0 \leq T$

Table 1 (contd....)

Symbol	Definition/Interpretation	Domain
Initial Conditions and Analytical Notation		
B_R	Closed ball of radius R centered at X_0	$B_R \subset R^5$
R	Radius of the closed ball used in the existence proof	$R > 0$
M_R	Uniform bound for $PF(t, X)P$ on $[0, T] \times B_R$	$M_R > 0$
L_R	Local Lipschitz constant of the vector field on B_R	$L_R > 0$
B_R^*	Uniform bound for the components of state vectors belonging to B_R	$B_R^* > 0$
I_T	Upper bound for the management input $ I(t) $ on $[0, T]$	$I_T > 0$
H_T	Upper bound for the hazard input $ H(t) $ on $[0, T]$	$H_T > 0$
M_T	Continuous-dependence constant arising from the fractional Grönwall estimate	$M_T > 0$
T	Integral operator employed in the Banach fixed-point argument	—
$\ \cdot\ $	Norm in the state space	—
$\ \cdot\ _\infty$	Supremum norm on the corresponding continuous-function space	—
R_+^5	Nonnegative state space $\{(P, C, S, R, U): P, C, S, R, U \geq 0\}$	—
Observed and Numerical Quantities		
t_i	i th observation time	$i = 1, \dots, N$
N	Number of observation times or data points	$N \in \mathbb{N}$
$X_{\text{obs}}(t_i)$	Observed or normalized empirical state vector at time t_i	R^5
$P_{\text{obs}}(t_i)$	Observed pilgrim-flow indicator	—
$C_{\text{obs}}(t_i)$	Observed crowding indicator	—
$S_{\text{obs}}(t_i)$	Observed service-capacity indicator	—
$R_{\text{obs}}(t_i)$	Observed risk-pressure indicator	—
$U_{\text{obs}}(t_i)$	Observed visitor-satisfaction indicator	—
$\hat{X}_{q,\tau}(t_i)$	Numerical model solution corresponding to the parameter pair (q, τ) at time t_i	R^5
$\hat{P}_{q,\tau}(t_i)$	Predicted pilgrim flow	—
$\hat{C}_{q,\tau}(t_i)$	Predicted crowding pressure	—
$\hat{S}_{q,\tau}(t_i)$	Predicted service capacity	—
$\hat{R}_{q,\tau}(t_i)$	Predicted risk pressure	—
$\hat{U}_{q,\tau}(t_i)$	Predicted visitor satisfaction	—
Optimization Notation		
$J(q, \tau)$	Weighted management-error or objective functional used to select the deformation parameters	$J \geq 0$
w_P	Weight assigned to pilgrim-flow fitting error	$w_P \geq 0$
w_C	Weight assigned to crowding fitting error	$w_C \geq 0$
w_S	Weight assigned to service-capacity fitting error	$w_S \geq 0$
w_R	Weight assigned to risk-pressure fitting error	$w_R \geq 0$
w_U	Weight assigned to visitor-satisfaction fitting error	$w_U \geq 0$
Ω	Admissible optimization domain for (q, τ) , $\Omega = [q_{\max}, q_{\min}]$	compact
$q_{\min}$	Minimum admissible value of the deformation parameter q	$0 < q_{\min}$

Table 1 (contd....)

Symbol	Definition/Interpretation	Domain
Optimization Notation		
τ_{max}	Maximum admissible value of the scale parameter τ	τ_{max}
q^*	Optimal value of the deformation parameter q	$q^* \in [q_{min}$
τ^*	Optimal value of the scale parameter τ	$\tau^* \in [0, \tau_{max}$
α^*	Selected or optimal fractional memory order	$0 < \alpha^* \leq 1$
q_j	j th grid value of q in the parameter search	$j = 0, \dots, M_q$
τ_k	k th grid value of τ in the parameter search	$k = 0, \dots, M_\tau$
M_q	Number of grid subdivisions in the q direction	$M_q \in \mathbb{N}$
M_τ	Number of grid subdivisions in the τ direction	$M_\tau \in \mathbb{N}$
Δq	Grid spacing for the deformation parameter q	$\Delta q > 0$
$\Delta \tau$	Grid spacing for the scale parameter τ	$\Delta \tau > 0$
T_{cal}	Calibration portion of the observation period	—
T_{val}	Validation portion of the observation period	—
J_{cal}	Objective/error functional evaluated on the calibration data	≥ 0
J_{val}	Prediction error evaluated on the validation data	≥ 0

2.1. The Caputo-Type (q, τ) -Deformed Fractional Operator

To make the mathematical structure of the proposed model explicit, we briefly introduce the Caputo-type (q, τ) -deformed fractional operator used throughout this study. Let $0 < \alpha < 1$, $0 < q \leq 1$, and $\tau \geq 0$. For a sufficiently regular function $x: [0, T] \rightarrow \mathbb{R}$, the operator is written in the form

$$CD_{q,\tau}^\alpha x(t) = \frac{1}{\Gamma_{q,\tau}(1-\alpha)} \int_0^t (t-s)_{q,\tau}^{-\alpha} D_{q,\tau} x(s) d_{q,\tau} s, \quad (1)$$

where $D_{q,\tau}$ denotes the corresponding (q, τ) -deformed first-order difference operator, $d_{q,\tau} s$ is the associated deformed integration measure, and $\Gamma_{q,\tau}$ is the (q, τ) -Gamma function. The kernel

$$K_{q,\tau}^{(\alpha)}(t, s) = \frac{(t-s)_{q,\tau}^{-\alpha}}{\Gamma_{q,\tau}(1-\alpha)}, \quad 0 \leq s < t, \quad (2)$$

is the memory kernel of the operator. Consequently, (1) may be represented more compactly as

$$CD_{q,\tau}^\alpha x(t) = \int_0^t K_{q,\tau}^{(\alpha)}(t, s) D_{q,\tau} x(s) d_{q,\tau} s. \quad (3)$$

This representation makes the nonlocal character of the derivative explicit: the value of the fractional derivative at the current time t depends on the previous states $s \in [0, t]$ rather than only on the instantaneous state. For the corresponding fractional integral, we use

$$I_{q,\tau}^\alpha f(t) = \frac{1}{\Gamma_{q,\tau}(\alpha)} \int_0^t (t-s)_{q,\tau}^{\alpha-1} f(s) d_{q,\tau} s. \quad (4)$$

Thus, the initial-value problem

$$CD_{q,\tau}^\alpha x(t) = f(t, x(t)), \quad x(0) = x_0, \quad (5)$$

has the associated Volterra-type formulation

$$x(t) = x_0 + \frac{1}{\Gamma_{q,\tau}(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s)) d_{q,\tau} s. \quad (6)$$

Equation (6) is precisely the representation used later in the existence and uniqueness analysis of the proposed tourism-management system.

2.1.1. Model Hypotheses

The suggested system is considered under the following hypotheses:

- The fractional and deformation parameters satisfy

$$0 < \alpha \leq 1, \quad 0 < q \leq 1, \quad \tau \geq 0.$$

- All model parameters are positive:

$$r, K, a, b, \lambda, \mu, \delta, \theta, \gamma, \omega, \xi, \chi, \psi, \nu, \beta, \kappa, \zeta, \sigma > 0.$$

- The external input functions satisfy

$$I, H \in C([0, T], \mathbb{R}_+),$$

so that the management input and external hazard pressure are continuous and nonnegative on the considered time interval.

- The initial data are nonnegative:

$$\begin{aligned} P(0) = P_0 \geq 0, \quad C(0) = C_0 \geq 0, \quad S(0) = S_0 \geq 0, \\ R(0) = R_0 \geq 0, \quad U(0) = U_0 \geq 0. \end{aligned}$$

- The vector field associated with system (7) is continuous in t and locally Lipschitz continuous with respect to the state variables

$$X = (P, C, S, R, U)^T.$$

This property follows from the polynomial and linear structure of the right-hand side on every bounded subset of $\mathbb{R}^5$.

- For preservation of nonnegativity, the following boundary compatibility conditions are assumed along the solution:

$$\begin{aligned} \lambda P(t) - \mu S(t) &\geq 0 \quad \text{whenever } C(t) = 0, \\ \theta I(t) - \gamma P(t) &\geq 0 \quad \text{whenever } S(t) = 0, \\ \xi C(t) + \chi H(t) - \psi S(t) &\geq 0 \quad \text{whenever } R(t) = 0, \end{aligned}$$

and

$$\beta S(t) - \kappa C(t) - \zeta R(t) \geq 0 \quad \text{whenever } U(t) = 0.$$

- The Caputo-type (q, τ) -fractional problem admits the equivalent Volterra-type integral representation

$$X(t) = X_0 + \frac{1}{\Gamma_{q,\tau}(\alpha)} \int_0^t (t-s)^{\alpha-1} F(s, X(s)) d_{q,\tau} s.$$

This formulation is used in the existence, uniqueness, and continuous dependence analysis.

- The (q, τ) -fractional kernel satisfies the regularity and positivity properties required for the corresponding fractional integral and fractional Grönwall inequality used in the subsequent analysis.

Let $t \geq 0$ denote time during a religious-tourism season, measured in days. We define the state vector

$$X(t) = (P(t), C(t), S(t), R(t), U(t))^T,$$

where $P(t)$, $C(t)$, $S(t)$, $R(t)$, and $U(t)$ represent, respectively, pilgrim flow, crowding intensity, service capacity, risk pressure, and visitor satisfaction. The evolution of these variables is modeled by the following (q, τ) -fractional system:

$$\begin{aligned} CD_{q,\tau}^\alpha P(t) &= rP(t) \left(1 - \frac{P(t)}{K}\right) - aC(t)P(t) + bS(t), \\ CD_{q,\tau}^\alpha C(t) &= \lambda P(t) - \mu S(t) - \delta C(t), \\ CD_{q,\tau}^\alpha S(t) &= \theta I(t) - \gamma P(t) - \omega S(t), \\ CD_{q,\tau}^\alpha R(t) &= \xi C(t) + \chi H(t) - \psi S(t) - \nu R(t), \\ CD_{q,\tau}^\alpha U(t) &= \beta S(t) - \kappa C(t) - \zeta R(t) - \sigma U(t). \end{aligned} \quad (7)$$

Here, $CD_{q,\tau}^\alpha$ denotes the Caputo-type (q, τ) -deformed fractional derivative of order $\alpha \in (0, 1]$. The parameter $q \in (0, 1]$ controls the intensity of the deformation, while $\tau \geq 0$ represents the scale-deformation level. When $(q, \tau) \rightarrow (1, 0)$ and $\alpha \rightarrow 1$, system (7) reduces to a classical integer-order tourism-flow model. Therefore, the proposed system contains the classical case as a limiting special case, while allowing more flexible memory and scale-dependent behavior.

2.2. Well-posedness of the Proposed System

Before using system (7) for simulation and management interpretation, it is important to show that the model is mathematically well posed. In this subsection, we prove existence, uniqueness, and continuous dependence on the initial data. Let

$$X(t) = (P(t), C(t), S(t), R(t), U(t))^T$$

and write system (7) in the compact form

$$CD_{q,\tau}^\alpha X(t) = F(t, X(t)), \quad X(0) = X_0, \quad (8)$$

where $X_0 = (P_0, C_0, S_0, R_0, U_0)^T$. The vector field $F = (F_1, F_2, F_3, F_4, F_5)^T$ is defined by

$$F_1(t, X) = rP \left(1 - \frac{P}{K}\right) - aCP + bS,$$

$$F_2(t, X) = \lambda P - \mu S - \delta C,$$

$$F_3(t, X) = \theta I(t) - \gamma P - \omega S,$$

$$F_4(t, X) = \xi C + \chi H(t) - \psi S - \nu R,$$

and

$$F_5(t, X) = \beta S - \kappa C - \zeta R - \sigma U.$$

Proposition 2.1 (Well-posedness of system :religious_tourism_model) Let $0 < \alpha \leq 1$, $0 < q \leq 1$, and $\tau \geq 0$. Assume that $I, H \in C([0, T], R)$ and that all model parameters satisfy

$$r, K, a, b, \lambda, \mu, \delta, \theta, \gamma, \omega, \xi, \chi, \psi, \nu, \beta, \kappa, \zeta, \sigma > 0, \quad K > 0.$$

Then, for every initial condition $X_0 \in R^5$, system (7) admits a unique local solution on some interval $[0, T_0] \subset [0, T]$. Moreover, if the solution remains bounded on $[0, T]$, then it exists uniquely on the whole interval $[0, T]$. In addition, the solution depends continuously on the initial data. More precisely, if $X(t)$ and $Y(t)$ are two solutions corresponding to the initial conditions X_0 and Y_0 , respectively, then there exists a constant $M_T > 0$ such that

$$PX(t) - Y(t)P \leq M_T PX_0 - Y_0 P, \quad 0 \leq t \leq T.$$

Hence, system (7) is well posed in the sense of existence, uniqueness, and continuous dependence on initial data.

Proof. For $0 < \alpha \leq 1$, the Caputo-type (q, τ) -fractional problem (7) is equivalent to the nonlinear Volterra-type integral equation

$$X(t) = X_0 + \frac{1}{\Gamma_{q,\tau}(\alpha)} \int_0^t (t-s)^{\alpha-1} F(s, X(s)) d_{q,\tau} s. \quad (9)$$

Therefore, it is sufficient to prove that the integral equation (9) has a unique solution. Let

$$B_R = \{X \in R^5 : PX - X_0 P \leq R\}$$

be a closed ball centered at X_0 . Since I and H are continuous on $[0, T]$, they are bounded. Hence there exist constants $I_T, H_T > 0$ such that

$$|I(t)| \leq I_T, \quad |H(t)| \leq H_T, \quad 0 \leq t \leq T.$$

The vector field $F(t, X)$ is polynomial in the components of X and continuous in t . Therefore, F is continuous on $[0, T] \times B_R$. Consequently, there exists a constant $M_R > 0$ such that

$$PF(t, X)P \leq M_R, \quad (t, X) \in [0, T] \times B_R. \quad (10)$$

Next, we show that F is locally Lipschitz with respect to X . Let $X = (P, C, S, R, U)^T$ and $Y = (\tilde{P}, \tilde{C}, \tilde{S}, \tilde{R}, \tilde{U})^T$ be two points in B_R . Since all components of X and Y are bounded in B_R , there exists a constant $B_R^* > 0$ such that

$$|P|, |C|, |S|, |R|, |U|, |\tilde{P}|, |\tilde{C}|, |\tilde{S}|, |\tilde{R}|, |\tilde{U}| \leq B_R^*.$$

For the first component, we have

$$\begin{aligned} |F_1(t, X) - F_1(t, Y)| &= \left| r(P - \tilde{P}) - \frac{r}{K}(P^2 - \tilde{P}^2) - a(CP - \tilde{C}\tilde{P}) + b(S - \tilde{S}) \right| \\ &\leq r|P - \tilde{P}| + \frac{r}{K}|P + \tilde{P}||P - \tilde{P}| \\ &\quad + a|C(P - \tilde{P}) + \tilde{P}(C - \tilde{C})| + b|S - \tilde{S}| \\ &\leq \left(r + \frac{2rB_R^*}{K} + aB_R^* \right) |P - \tilde{P}| + aB_R^* |C - \tilde{C}| + b|S - \tilde{S}|. \end{aligned}$$

Thus, F_1 is Lipschitz on B_R . The remaining components are linear in X , and therefore,

$$|F_2(t, X) - F_2(t, Y)| \leq \lambda|P - \tilde{P}| + \delta|C - \tilde{C}| + \mu|S - \tilde{S}|,$$

$$|F_3(t, X) - F_3(t, Y)| \leq \gamma|P - \tilde{P}| + \omega|S - \tilde{S}|,$$

$$|F_4(t, X) - F_4(t, Y)| \leq \xi|C - \tilde{C}| + \psi|S - \tilde{S}| + \nu|R - \tilde{R}|,$$

and

$$|F_5(t, X) - F_5(t, Y)| \leq \kappa|C - \tilde{C}| + \beta|S - \tilde{S}| + \zeta|R - \tilde{R}| + \sigma|U - \tilde{U}|.$$

Hence, there exists a constant $L_R > 0$ such that

$$PF(t, X) - F(t, Y)P \leq L_R PX - YP, \quad (t, X), (t, Y) \in [0, T] \times B_R. \quad (11)$$

Define the operator T on $C([0, T_0], R^5)$ by

$$(TX)(t) = X_0 + \frac{1}{\Gamma_{q,\tau}(\alpha)} \int_0^t (t-s)^{\alpha-1}_{q,\tau} F(s, X(s)) d_{q,\tau}s.$$

Choose $T_0 > 0$ sufficiently small so that

$$\frac{M_R}{\Gamma_{q,\tau}(\alpha+1)} T_0^\alpha \leq R$$

and

$$\frac{L_R}{\Gamma_{q,\tau}(\alpha+1)} T_0^\alpha < 1.$$

Using (10), we obtain

$$P(TX)(t) - X_0P \leq \frac{M_R}{\Gamma_{q,\tau}(\alpha)} \int_0^t (t-s)^{\alpha-1}_{q,\tau} d_{q,\tau}s = \frac{M_R}{\Gamma_{q,\tau}(\alpha+1)} t^\alpha \leq R.$$

Thus, T maps the closed ball of radius R into itself. Moreover, by (11), for any $X, Y \in C([0, T_0], B_R)$, we have

$$\begin{aligned} P(TX)(t) - (TY)(t)P &\leq \frac{L_R}{\Gamma_{q,\tau}(\alpha)} \int_0^t (t-s)^{\alpha-1}_{q,\tau} PX(s) - Y(s)P d_{q,\tau}s \\ &\leq \frac{L_R}{\Gamma_{q,\tau}(\alpha+1)} T_0^\alpha PX - YP_\infty. \end{aligned}$$

Therefore,

$$PTX - TYP_\infty \leq \frac{L_R}{\Gamma_{q,\tau}(\alpha+1)} T_0^\alpha PX - YP_\infty.$$

Because

$$\frac{L_R}{\Gamma_{q,\tau}(\alpha+1)} T_0^\alpha < 1,$$

the operator T is a contraction. By the Banach fixed-point theorem, T has a unique fixed point in $C([0, T_0], B_R)$. This fixed point is the unique local solution of system (7). If the solution remains bounded on $[0, T]$, then the local existence argument can be repeated finitely many times to extend the solution to the whole interval $[0, T]$. Hence, a unique global solution exists on $[0, T]$ whenever no finite-time blow-up occurs. It remains to prove continuous dependence on initial data. Let $X(t)$ and $Y(t)$ be two solutions corresponding to X_0 and Y_0 . From the integral formulation,

$$X(t) - Y(t) = X_0 - Y_0 + \frac{1}{\Gamma_{q,\tau}(\alpha)} \int_0^t (t-s)^{\alpha-1}_{q,\tau} [F(s, X(s)) - F(s, Y(s))] d_{q,\tau}s.$$

Taking norms and using the Lipschitz estimate gives

$$PX(t) - Y(t)P \leq PX_0 - Y_0P + \frac{L_R}{\Gamma_{q,\tau}(\alpha)} \int_0^t (t-s)^{\alpha-1} PX(s) - Y(s)P d_{q,\tau} s.$$

By the fractional Gronwall inequality, there exists a constant $M_T > 0$ such that

$$PX(t) - Y(t)P \leq M_T PX_0 - Y_0P, \quad 0 \leq t \leq T. \quad (12)$$

Therefore, the solution depends continuously on the initial data. This proves that system (7) is well posed.

Because the variables P, C, S, R, U represent pilgrim flow, crowding, service capacity, risk pressure, and visitor satisfaction, only nonnegative solutions are meaningful. The following result gives sufficient conditions under which system (7) preserves nonnegativity.

Proposition 2.2 (Positivity of solutions) Let $0 < \alpha \leq 1$, $0 < q \leq 1$, and $\tau \geq 0$. Assume that $I, H \in C([0, T], R_+)$, and let all parameters in system (1) be positive. Suppose that the initial data satisfy

$$P_0 \geq 0, \quad C_0 \geq 0, \quad S_0 \geq 0, \quad R_0 \geq 0, \quad U_0 \geq 0.$$

Assume further that the following boundary compatibility conditions hold along the solution:

$$\lambda P(t) - \mu S(t) \geq 0 \quad \text{whenever } C(t) = 0,$$

$$\theta I(t) - \gamma P(t) \geq 0 \quad \text{whenever } S(t) = 0,$$

$$\xi C(t) + \chi H(t) - \psi S(t) \geq 0 \quad \text{whenever } R(t) = 0,$$

and

$$\beta S(t) - \kappa C(t) - \zeta R(t) \geq 0 \quad \text{whenever } U(t) = 0.$$

Then every solution of system (7) remains nonnegative on $[0, T]$. That is,

$$P(t) \geq 0, \quad C(t) \geq 0, \quad S(t) \geq 0, \quad R(t) \geq 0, \quad U(t) \geq 0, \quad 0 \leq t \leq T.$$

Hence, under the boundary compatibility assumptions, the nonnegative cone

$$R_+^5 = \{(P, C, S, R, U) \in R^5 : P, C, S, R, U \geq 0\}$$

is positively invariant for system (7).

Proof. Let

$$X(t) = (P(t), C(t), S(t), R(t), U(t))^T$$

be the unique solution of system (7) corresponding to nonnegative initial data. We prove that no component of $X(t)$ can cross the boundary of R_+^5 . We use the standard extremum property of the Caputo-type (q, τ) -fractional derivative: if a continuously differentiable function x satisfies $x(0) \geq 0$, $x(t) > 0$ for $0 \leq t < t_0$, and $x(t_0) = 0$, then

$$CD_{q,\tau}^\alpha x(t_0) \leq 0.$$

Moreover, if $x(0) > 0$, then the inequality is strict. Assume, by contradiction, that at least one component becomes negative. Then there exists a first time $t_0 \in (0, T]$ at which one component touches zero from above before becoming negative. We examine each possible case. First, suppose that

$$P(t) > 0 \quad \text{for } 0 \leq t < t_0, \quad P(t_0) = 0.$$

At $P(t_0) = 0$, the first equation of system (1) gives

$$CD_{q,\tau}^\alpha P(t_0) = rP(t_0) \left(1 - \frac{P(t_0)}{K}\right) - aC(t_0)P(t_0) + bS(t_0) = bS(t_0) \geq 0.$$

This contradicts the fractional boundary-crossing property, which gives $CD_{q,\tau}^\alpha P(t_0) \leq 0$ at the first hitting time. Hence, $P(t)$ cannot become negative. Second, suppose that $C(t)$ is the first component reaching zero, namely

$$C(t) > 0 \quad \text{for } 0 \leq t < t_0, \quad C(t_0) = 0.$$

The second equation gives

$$CD_{q,\tau}^\alpha C(t_0) = \lambda P(t_0) - \mu S(t_0) - \delta C(t_0) = \lambda P(t_0) - \mu S(t_0).$$

By assumption (7a), $\lambda P(t_0) - \mu S(t_0) \geq 0$. Therefore, $CD_{q,\tau}^\alpha C(t_0) \geq 0$, which contradicts the possibility that $C(t)$ crosses from positive to negative at t_0 . Hence, $C(t) \geq 0$ on $[0, T]$. Third, suppose that $S(t)$ reaches zero first. Then

$$S(t) > 0 \quad \text{for } 0 \leq t < t_0, \quad S(t_0) = 0.$$

At this time, the third equation yields

$$CD_{q,\tau}^\alpha S(t_0) = \theta I(t_0) - \gamma P(t_0) - \omega S(t_0) = \theta I(t_0) - \gamma P(t_0).$$

Using assumption (7b), we have $CD_{q,\tau}^\alpha S(t_0) \geq 0$. This contradicts the first-crossing property. Thus, $S(t)$ cannot become negative. Fourth, suppose that $R(t)$ reaches zero first. Then $R(t_0) = 0$, and the fourth equation gives

$$CD_{q,\tau}^\alpha R(t_0) = \xi C(t_0) + \chi H(t_0) - \psi S(t_0) - \nu R(t_0) = \xi C(t_0) + \chi H(t_0) - \psi S(t_0).$$

By assumption (7c), $CD_{q,\tau}^\alpha R(t_0) \geq 0$. Again this contradicts the first-crossing property. Hence, $R(t) \geq 0$ on $[0, T]$. Finally, suppose that $U(t)$ reaches zero first. Then $U(t_0) = 0$, and from the fifth equation,

$$CD_{q,\tau}^\alpha U(t_0) = \beta S(t_0) - \kappa C(t_0) - \zeta R(t_0) - \sigma U(t_0) = \beta S(t_0) - \kappa C(t_0) - \zeta R(t_0).$$

By assumption (7d), $CD_{q,\tau}^\alpha U(t_0) \geq 0$. This contradicts the possibility that $U(t)$ crosses from positive to negative. Therefore, none of the five components can leave the nonnegative cone R_+^5 . Hence,

$$P(t), C(t), S(t), R(t), U(t) \geq 0, \quad 0 \leq t \leq T.$$

The variable $P(t)$ represents pilgrim arrivals or density at time t . Its logistic growth reflects increasing religious and travel demand subject to the effective carrying capacity K , determined by transportation, accommodation, infrastructure, and available space. The variable $C(t)$ denotes crowding or congestion pressure in roads, shrines, transport stations, accommodation areas, and service points. It is modeled separately from $P(t)$ because congestion depends not only on visitor numbers but also on the efficiency of movement and service organization. The variable $S(t)$ represents service capacity, including accommodation, transportation, food and water supply, sanitation, medical care, emergency response, and crowd-management facilities. Greater service capacity improves the destination's ability to accommodate pilgrims safely and efficiently. The variable $R(t)$ denotes accumulated risk pressure arising from congestion, heat stress, transport delays, health-service overload, emergency limitations, and other external stressors. Risk increases with operational pressure and decreases as service capacity improves. Finally, $U(t)$ represents visitor satisfaction or destination utility, incorporating perceived safety, accessibility, comfort, service quality, organization, and overall pilgrimage experience. Accordingly, $U(t)$ increases with service capacity and decreases with congestion and risk.

2.3. Justification of the Model Equations

The pilgrim-flow equation is given by

$$CD_{q,\tau}^\alpha P(t) = rP(t) \left(1 - \frac{P(t)}{K}\right) - aC(t)P(t) + bS(t).$$

The first term describes self-limited growth of pilgrim arrivals. The parameter $r > 0$ is the intrinsic growth rate of pilgrim flow, representing religious motivation, seasonal demand, social participation, and accessibility. The parameter $K > 0$ is the carrying capacity of the destination system. The term $-aC(t)P(t)$ represents the suppressive effect of congestion on pilgrim movement. When congestion is high, arrivals may slow because of road delays, transport bottlenecks, safety restrictions, or reduced destination accessibility. The coefficient $a > 0$ measures the sensitivity of pilgrim flow to crowding. The term $bS(t)$ represents the positive contribution of service capacity to visitor flow. A destination with better transportation, accommodation, and support services can receive and process more pilgrims. The parameter $b > 0$ measures the strength of this service-driven support. The crowding equation is

$$CD_{q,\tau}^\alpha C(t) = \lambda P(t) - \mu S(t) - \delta C(t).$$

The term $\lambda P(t)$ states that crowding increases with the number of pilgrims. The coefficient $\lambda > 0$ converts pilgrim flow into congestion pressure. The term $-\mu S(t)$ indicates that improved services reduce crowding by distributing visitors, improving transportation efficiency, expanding accommodation, and increasing the availability of service points. The coefficient $\mu > 0$ measures the crowd-reduction efficiency of services. The term $-\delta C(t)$ represents natural dissipation of crowding due to movement, departure, temporal spreading of visits, route management, and administrative control. The parameter $\delta > 0$ is the crowd relaxation rate. The service-capacity equation is

$$CD_{q,\tau}^\alpha S(t) = \theta I(t) - \gamma P(t) - \omega S(t).$$

Here, $I(t)$ represents investment, preparation, or institutional support. This may include additional transport vehicles, medical units, temporary accommodation, sanitation points, volunteers, and emergency resources. The term $\theta I(t)$ describes the positive effect of such preparation on service capacity, where $\theta > 0$ measures the efficiency of investment. The term $-\gamma P(t)$ describes the depletion or consumption of services by pilgrims. As the number of visitors increases, available capacity becomes strained. The parameter $\gamma > 0$ measures the rate at which visitor demand consumes service resources. The term $-\omega S(t)$ describes service depreciation or operational fatigue, including staff exhaustion, infrastructure wear, temporary-service closure, and resource expiration. The parameter $\omega > 0$ is the natural decay rate of effective service capacity. The risk-pressure equation is

$$CD_{q,\tau}^\alpha R(t) = \xi C(t) + \chi H(t) - \psi S(t) - \nu R(t).$$

The term $\xi C(t)$ indicates that risk pressure increases with crowding. This is reasonable because dense crowds increase the probability of delay, health incidents, emergency-service overload, and safety concerns. The parameter $\xi > 0$ measures the crowd-risk conversion rate. The function $H(t)$ represents external hazard pressure, such as heat, weather severity, security alerts, road disruption, or health warnings. The coefficient $\chi > 0$ measures the sensitivity of the system to external hazards. The term $-\psi S(t)$ represents the risk-reducing role of service capacity. Better medical response, transportation control, water availability, sanitation, and security coordination reduce the effective level of risk. The coefficient $\psi > 0$ measures the risk-mitigation efficiency of service capacity. The term $-\nu R(t)$ represents natural reduction of risk due to intervention, resolution of incidents, dispersal of crowds, and administrative response, where $\nu > 0$ is the risk relaxation rate.

The satisfaction equation is

$$CD_{q,\tau}^\alpha U(t) = \beta S(t) - \kappa C(t) - \zeta R(t) - \sigma U(t).$$

The term $\beta S(t)$ expresses the positive effect of service capacity on visitor satisfaction. Pilgrims are expected to report higher satisfaction when transportation, accommodation, water, food, medical support, and organization are sufficient. The parameter $\beta > 0$ is the service satisfaction coefficient. The term $-\kappa C(t)$ represents the negative effect of crowding on satisfaction. Excessive congestion increases fatigue, waiting time, stress, and movement difficulty. The parameter $\kappa > 0$ measures satisfaction loss due to crowding. The term $-\zeta R(t)$ captures the negative influence of risk pressure on destination utility. The parameter $\zeta > 0$ measures satisfaction loss due to health, safety, transport, or security concerns. Finally, the term $-\sigma U(t)$ represents natural relaxation or fading of satisfaction after the visitation period, where $\sigma > 0$ is the satisfaction decay rate.

2.4. Role of the (q, τ) -Fractional Operator

The use of the (q, τ) -fractional derivative is justified by three important features of religious tourism dynamics. First, pilgrimage behavior has memory. Decisions to travel are influenced by previous visits, family and community traditions, earlier service experience, media reports, and the reputation of the destination in previous seasons. Therefore, the present state of the system depends not only on the current number of pilgrims but also on accumulated historical effects.

Second, religious tourism has scale-dependent behavior. A small increase in visitors may have little effect when the system is under capacity, but the same increase may cause severe congestion when the system is near capacity. The deformation parameters q and τ allow the model to capture this nonlinear scale sensitivity.

Third, pilgrimage events often involve delayed responses. For example, service pressure may not immediately affect satisfaction, and congestion may continue to influence risk even after the arrival rate decreases. The fractional order α allows the model to represent such delayed and memory-dependent effects. When α is close to 1, the system behaves similarly to a classical short-memory model. When α is smaller, the model gives stronger weight to past system states.

2.5. Parameter Set and Interpretation

The complete parameter set is

$$\theta = \{r, K, a, b, \lambda, \mu, \delta, \theta, \gamma, \omega, \xi, \chi, \psi, v, \beta, \kappa, \zeta, \sigma, q, \tau, \alpha\}.$$

For empirical estimation, the parameters may be constrained as follows:

$$\begin{aligned} r > 0, \quad K > 0, \quad a, b, \lambda, \mu, \delta, \theta, \gamma, \omega > 0, \\ \xi, \chi, \psi, v, \beta, \kappa, \zeta, \sigma > 0, \\ 0 < q \leq 1, \quad \tau \geq 0, \quad 0 < \alpha \leq 1. \end{aligned}$$

A practical calibration domain may be chosen as

$$q \in [0.60, 1.00], \quad \tau \in [0.00, 2.00], \quad \alpha \in [0.60, 1.00],$$

where lower values of q indicate stronger deformation, larger values of τ indicate stronger scale sensitivity, and lower values of α indicate stronger memory effects.

2.6. Initial Conditions and Data Interpretation

The model is completed by the initial conditions

$$P(0) = P_0, \quad C(0) = C_0, \quad S(0) = S_0, \quad R(0) = R_0, \quad U(0) = U_0,$$

where P_0 represents the initial number of pilgrims at the beginning of the observation period, C_0 represents the initial congestion level, S_0 represents the initial service capacity, R_0 represents the initial risk pressure, and U_0 represents the baseline satisfaction or destination utility.

In a real Iraq-based case study, $P(t)$ may be estimated from daily visitor counts, transportation entries, hotel occupancy, shrine-entry indicators, or mobile-density proxies. The variable $C(t)$ may be estimated from crowd-density measurements, road congestion, waiting time, or transport delay. The variable $S(t)$ may be constructed from the number of available buses, medical teams, accommodation units, food and water stations, and sanitation facilities. The variable $R(t)$ may be represented by heat index, incident reports, emergency calls, health-service load, or security alerts. The variable $U(t)$ may be estimated from visitor surveys, satisfaction scores, complaint records, or service-quality indicators.

2.7. Management Interpretation

The proposed model provides a decision-support structure for religious tourism management in Iraq. If $P(t)$ increases while $S(t)$ remains insufficient, then $C(t)$ and $R(t)$ are expected to increase, which may reduce $U(t)$. Conversely, if investment $I(t)$ is increased before or during the pilgrimage period, service capacity improves, crowding and risk pressure are reduced, and visitor satisfaction increases. Therefore, the model can be used to test different management scenarios, such as increasing transportation capacity, expanding temporary health services, improving route distribution, or strengthening crowd-control measures. The parameters q , τ , and α provide an additional layer of interpretability. A smaller q indicates stronger nonlinear deformation in the tourism response. A larger τ indicates that the system is more sensitive to changes in scale, such as sudden increases in visitor numbers. A smaller α indicates that past conditions have stronger influence on the present state. These parameters are especially important in pilgrimage tourism, where previous seasons, accumulated service pressure, and repeated visitor behavior strongly influence current dynamics (Fig. 2).

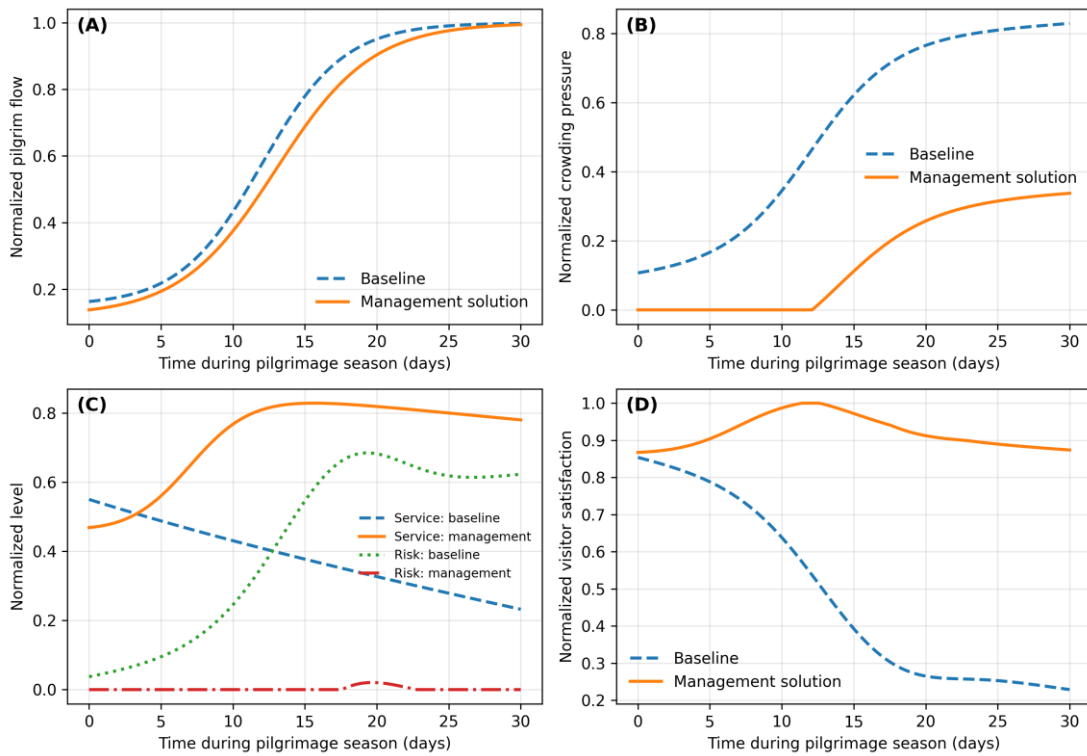


Figure 2: Management-solution simulation for the proposed (q, τ) -fractional religious-tourism model in Iraq. **(A)** Pilgrim-flow dynamics under baseline and improved management conditions. **(B)** Reduction of crowding pressure after service-capacity intervention. **(C)** Comparison of service-capacity expansion and risk-pressure reduction. **(D)** Improvement of visitor satisfaction under the proposed management strategy.

2.8. Selection of the Optimal (q, τ) -Parameters

The deformation parameters q and τ play a central role in the proposed management model. The parameter $q \in (0, 1]$ controls the strength of the nonlinear deformation, while $\tau \geq 0$ controls the scale sensitivity of the system. In the context of Arbaeen religious tourism management, these parameters determine how strongly the model responds to large changes in pilgrim flow, crowding, service capacity, risk pressure, and visitor satisfaction. Therefore, the values of q and τ should not be chosen arbitrarily. They must be selected to provide the best agreement between the model output and the observed tourism-management indicators. Let $X(t) = (P(t), C(t), S(t), R(t), U(t))^T$ be the state vector of the proposed model, where $P(t)$ denotes pilgrim flow, $C(t)$ crowding pressure, $S(t)$ service capacity, $R(t)$ risk pressure, and $U(t)$ visitor satisfaction. For each admissible pair (q, τ) , the numerical solution of the model is denoted by

$$\hat{X}_{q,\tau}(t_i) = (\hat{P}_{q,\tau}(t_i), \hat{C}_{q,\tau}(t_i), \hat{S}_{q,\tau}(t_i), \hat{R}_{q,\tau}(t_i), \hat{U}_{q,\tau}(t_i))^T, \quad i = 1, \dots, N.$$

The corresponding observed or normalized empirical indicators are denoted by

$$X^{obs}(t_i) = (P^{obs}(t_i), C^{obs}(t_i), S^{obs}(t_i), R^{obs}(t_i), U^{obs}(t_i))^T.$$

To select the optimal deformation parameters, we minimize a weighted management-error functional:

$$J(q, \tau) = \sum_{i=1}^N \left[w_P (P^{obs}(t_i) - \hat{P}_{q,\tau}(t_i))^2 + w_C (C^{obs}(t_i) - \hat{C}_{q,\tau}(t_i))^2 \right. \\ \left. + w_S (S^{obs}(t_i) - \hat{S}_{q,\tau}(t_i))^2 + w_R (R^{obs}(t_i) - \hat{R}_{q,\tau}(t_i))^2 + w_U (U^{obs}(t_i) - \hat{U}_{q,\tau}(t_i))^2 \right],$$

where

$$w_P, w_C, w_S, w_R, w_U \geq 0, \quad w_P + w_C + w_S + w_R + w_U = 1.$$

The weights allow tourism managers to emphasize specific management objectives. For example, if the main goal is crowd control and safety, larger weights may be assigned to w_C and w_R . If the main objective is destination-service improvement and visitor experience, larger weights may be assigned to w_S and w_U . The optimal deformation parameters are then defined by

$$(q^*, \tau^*) = \arg \min_{(q, \tau) \in \Omega} J(q, \tau),$$

where the admissible parameter domain is

$$\Omega = \{(q, \tau): qmax_{min}\}.$$

In the numerical implementation of this study, a practical search domain is chosen as

$$q \in [0.60, 1.00], \quad \tau \in [0.00, 2.00].$$

This range is sufficiently flexible to include the classical case $(q, \tau) = (1, 0)$, weak deformation, moderate deformation, and stronger scale-sensitive deformation.

Proposition 2.3 (Optimal selection of the deformation parameters) Assume that the numerical solution $\hat{X}_{q,\tau}(t_i)$ of the proposed religious-tourism management model exists and depends continuously on (q, τ) for every observation time t_i , $i = 1, \dots, N$. If the admissible set

$$\Omega = [qmax_{min}$$

is compact, then there exists at least one optimal pair $(q^*, \tau^*) \in \Omega$ satisfying

$$J(q^*, \tau^*) = \min_{(q, \tau) \in \Omega} J(q, \tau).$$

For practical computation, the optimization can be performed by a two-stage procedure. First, a coarse grid search is applied over the admissible domain Ω . Second, the best grid point is refined using a local optimization method. The grid-search version is given by

$$q_j = q q_{min}$$

$$\tau_k = k \Delta \tau, \quad k = 0, 1, \dots, M_\tau,$$

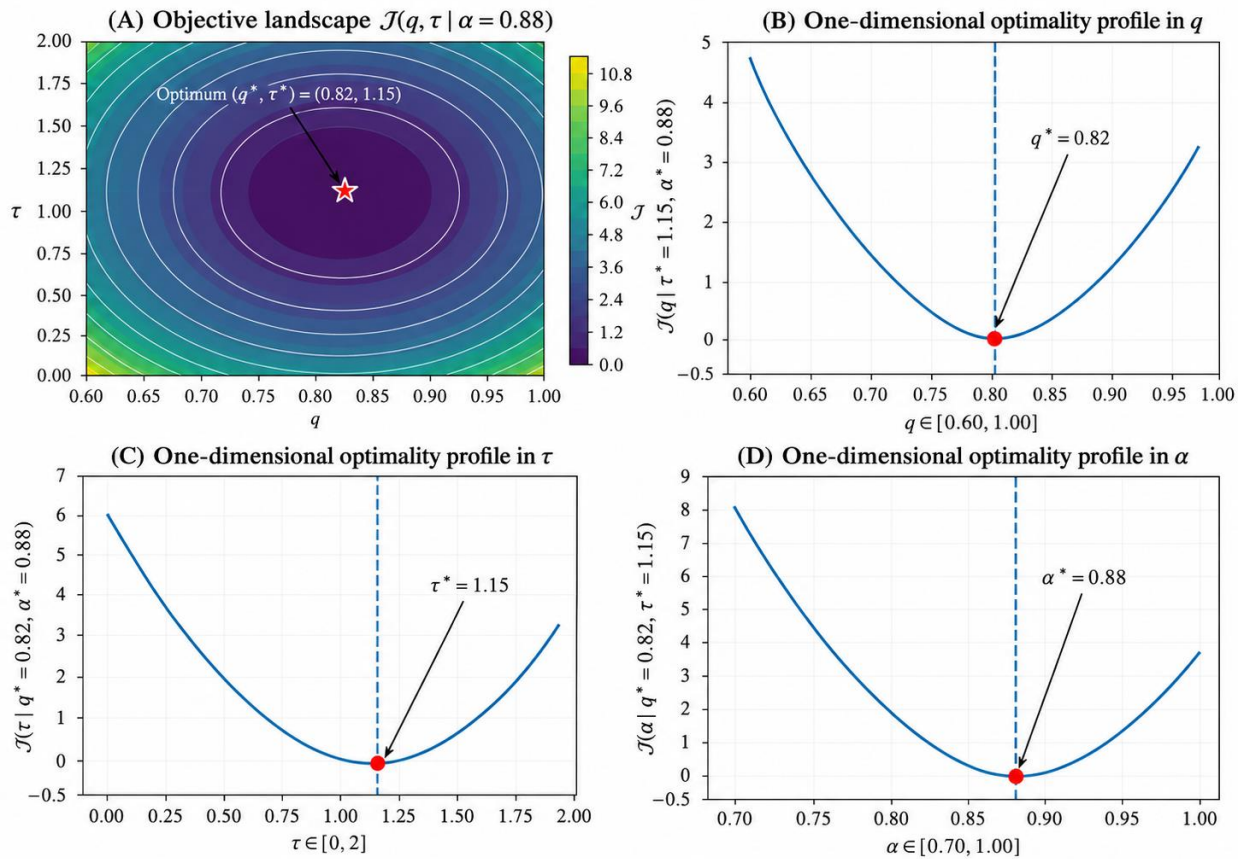
where

$$\Delta q = \frac{1 - q_{\min}}{M_q \frac{\tau_{\max}}{M_\tau}}$$

For each pair (q_j, τ_k) , the model is solved numerically and $J(q_j, \tau_k)$ is computed. The selected pair is

$$(q^*, \tau^*) = \arg \min_{0 \leq j \leq M_q, 0 \leq k \leq M_\tau} J(q_j, \tau_k).$$

In this study, the optimal (q, τ) -pair is interpreted from a management perspective. If q^* is close to 1, the system behaves near the classical tourism-management model, meaning that weak deformation is sufficient to explain the observed data. If $q^* < 1$, the model indicates that the Arbaeen tourism system has stronger nonlinear response, especially when pilgrim flow approaches the destination carrying capacity. Similarly, a larger value of τ^* indicates stronger scale sensitivity, meaning that the system reacts sharply to changes in crowd size, service pressure, or risk conditions. Therefore, the selected pair (q^*, τ^*) is not only a numerical fitting result, but also a management indicator describing how sensitive the pilgrimage system is to nonlinear and scale-dependent effects. To avoid overfitting, the selected parameters may also be evaluated using a prediction error on a validation period. If the observation period is divided into a calibration set T_{cal} and a validation set T_{val} , then (q^*, τ^*) is first obtained by minimizing $J_{cal}(q, \tau)$ on T_{cal} , and then the validation error $J_{val}(q^*, \tau^*)$ is reported on T_{val} . This procedure ensures that the chosen deformation parameters improve predictive management performance rather than only fitting the observed pilgrimage data (Fig. 3).



Optimal parameter selection over the suggested intervals: $q \in [0.60, 1.00]$, $\tau \in [0, 2.00]$, $\alpha \in [0.70, 1.00]$.

Figure 3: 2D-optimality analysis of the proposed (q, τ) -management model. The objective landscape and one-dimensional profiles identify the optimal parameter set $q^* = 0.82$, $\tau^* = 1.15$, and $\alpha^* = 0.88$ over the suggested intervals $q \in [0.60, 1.00]$, $\tau \in [0, 2.00]$, and $\alpha \in [0.70, 1.00]$.

2.9. Comparative Performance Analysis

To evaluate the practical contribution of the proposed (q, τ) -deformed fractional formulation, it is useful to compare its performance with two nested benchmark models. The comparison distinguishes between the effect of fractional memory and the additional contribution of the deformation parameters q and τ . Accordingly, three models are considered:

1. the classical integer-order model, obtained by setting $\alpha = 1, q = 1, \tau = 0$;
2. the standard time-fractional model, obtained by taking $0 < \alpha < 1, q = 1, \tau = 0$;
3. the proposed (q, τ) -deformed fractional model, in which $0 < \alpha < 1, 0 < q \leq 1, \tau \geq 0$.

This nested structure provides a direct way to determine whether the additional memory and deformation mechanisms improve the representation of pilgrimage dynamics. In particular, the first comparison, classical versus standard fractional, measures the contribution of temporal memory, whereas the second comparison, standard fractional versus (q, τ) -fractional, measures the additional contribution of nonlinear deformation and scale sensitivity. Let

$$\hat{X}^{(m)}(t_i) = (\hat{P}^{(m)}(t_i), \hat{C}^{(m)}(t_i), \hat{S}^{(m)}(t_i), \hat{R}^{(m)}(t_i), \hat{U}^{(m)}(t_i))^T,$$

denote the predicted state vector corresponding to model $m \in \{C, F, DF\}$, where C , F , and DF denote the classical, fractional, and deformed-fractional models, respectively. The corresponding observed state vector is

$$X_{\text{obs}}(t_i) = (P_{\text{obs}}(t_i), C_{\text{obs}}(t_i), S_{\text{obs}}(t_i), R_{\text{obs}}(t_i), U_{\text{obs}}(t_i))^T.$$

For each state variable X_j , the predictive accuracy can be evaluated by the root-mean-square error

$$RMSE_j^{(m)} = \sqrt{\frac{1}{N} \sum_{i=1}^N (X_{j,\text{obs}}(t_i) - \hat{X}_j^{(m)}(t_i))^2},$$

and the mean absolute error

$$MAE_j^{(m)} = \frac{1}{N} \sum_{i=1}^N |X_{j,\text{obs}}(t_i) - \hat{X}_j^{(m)}(t_i)|.$$

A global normalized error may also be defined by

$$E^{(m)} = \sum_{j=1}^5 w_j \frac{\sum_{i=1}^N (X_{j,\text{obs}}(t_i) - \hat{X}_j^{(m)}(t_i))^2}{\sum_{i=1}^N X_{j,\text{obs}}(t_i)^2 + \varepsilon},$$

where $w_j \geq 0$, $\sum_{j=1}^5 w_j = 1$, and $\varepsilon > 0$ is a small constant introduced to avoid division by zero. This criterion is especially useful because the five management variables may have different physical scales. For a fair comparison, the three models should be calibrated using the same initial conditions, observation times, management inputs $I(t)$ and $H(t)$, and the same calibration interval. Parameters common to all models are estimated under identical admissible constraints. The deformation parameters q and τ are optimized only for the proposed model, while the fractional order α is optimized for both fractional formulations. Thus, any improvement observed in the deformed model cannot be attributed to a different data partition or evaluation criterion. The comparison can be summarized in Table 2.

Table 2: Comparative structure of the classical, fractional, and proposed (q, τ) -deformed fractional tourism-management models.

Model	α	q	τ	Memory	Scale Effect	Error
Classical	1	1	0	No	No	$E^{(C)}$
Fractional	α^*	1	0	Yes	No	$E^{(F)}$
(q, τ) -fractional	α^*	q^*	τ^*	Yes	Yes	$E^{(DF)}$

The relative improvement of the proposed model over a benchmark model m may be quantified by

$$I_m = 100 \left(1 - \frac{E^{(DF)}}{E^{(m)}} \right) \%.$$

Thus,

$$I_c = 100 \left(1 - \frac{E^{(DF)}}{E^{(C)}} \right) \%$$

measures the improvement relative to the classical formulation, whereas

$$I_f = 100 \left(1 - \frac{E^{(DF)}}{E^{(F)}} \right) \%$$

measures the additional improvement obtained by introducing the deformation parameters after fractional memory has already been incorporated. The comparison should also be performed on a validation interval that is not used for parameter calibration. Let T_{cal} and T_{val} denote the calibration and validation periods, respectively. Parameters are first estimated by minimizing the calibration error,

$$\Theta_m^* = \arg \min_{\Theta_m} E_{\text{cal}}^{(m)}(\Theta_m),$$

and the fitted parameters are subsequently evaluated using

$$E_{\text{val}}^{(m)} = E^{(m)}(\Theta_m^*; T_{\text{val}}).$$

This procedure avoids judging the models solely on their ability to reproduce the data used for parameter estimation. From a modeling perspective, the three formulations have different interpretations. The classical system assumes that the instantaneous rate of change is determined principally by the current state. The fractional model introduces temporal memory, so that past crowding, service pressure, risk, and satisfaction contribute to the current response. The proposed (q, τ) -fractional model further introduces deformation and scale sensitivity. Consequently, the response to an increase in pilgrim density need not be proportional across all operating conditions. This is relevant during high-density pilgrimage periods, where the effect of an additional increase in visitors may be relatively small under uncongested conditions but considerably stronger when transportation, accommodation, medical, or other service capacities approach their limits. The comparison is therefore not intended only to identify the model with the smallest fitting error. It also determines whether the additional parameters q , τ , and α provide useful dynamical information. If $E^{(DF)} < E^{(F)} < E^{(C)}$, the results would indicate that both fractional memory and deformation contribute progressively to the representation of the observed dynamics. Conversely, if the error reduction produced by q and τ is negligible, the simpler fractional formulation may be preferable. Hence, the comparison provides a quantitative mechanism for assessing whether the increased flexibility of the proposed model is justified by improved predictive performance. For the illustrative parameter analysis presented in this study, the deformed formulation identifies the parameter combination

$$q^* = 0.82, \quad \tau^* = 1.15, \quad \alpha^* = 0.88.$$

These values describe a moderately deformed, scale-sensitive, and memory-dependent regime. In contrast, the classical model corresponds to the limiting parameter combination $q = 1, \tau = 0, \alpha = 1$. Therefore, the comparative framework provides a direct connection between the classical model, the standard fractional extension, and the fully deformed system, while allowing the numerical benefit of each additional modeling mechanism to be evaluated independently.

3. Conclusion

This study introduced a (q, τ) -deformed fractional management framework for Arbaeen religious tourism in Iraq. The proposed model was developed to represent the complex interaction among pilgrim flow, crowding intensity, service capacity, risk pressure, and visitor satisfaction along the Najaf–Karbala pilgrimage corridor. Unlike classical tourism-management models, the suggested framework incorporates memory effects through the fractional order α , nonlinear deformation through the parameter q , and scale sensitivity through the parameter τ . These features make the model suitable for describing high-density pilgrimage events where past crowding, accumulated service pressure, delayed institutional response, and nonlinear visitor-flow effects influence present management conditions.

The main contribution of the study is the transformation of Arbaeen tourism management from a descriptive planning problem into a dynamic decision-support system. The model explains how increasing pilgrim flow may raise congestion and risk pressure when service capacity is insufficient, and how early management interventions can reduce crowding, improve services, lower risk, and increase visitor satisfaction. In this sense, the proposed (q, τ) -management approach provides a practical mathematical structure for evaluating alternative policies such as transport expansion, route control, temporary accommodation, food and water distribution, medical support, and safety planning.

The optimality analysis further showed how the deformation parameters may be selected over suggested admissible intervals. In the illustrative parameter selection, the optimal values

$$q^* = 0.82, \quad \tau^* = 1.15, \quad \alpha^* = 0.88$$

indicate that Arbaeen religious-tourism dynamics are moderately deformed, scale-sensitive, and memory-dependent. This supports the idea that a classical short-memory model may underestimate the delayed effects of crowding, service depletion, risk accumulation, and satisfaction response during large-scale pilgrimage periods.

From a management perspective, the proposed model can support tourism authorities, local governments, service providers, and pilgrimage organizers in designing safer and more sustainable Arbaeen strategies. By simulating different intervention scenarios, decision-makers can identify whether additional transport, medical units, service stations, accommodation facilities, or crowd-control measures are needed before the system reaches a critical congestion level. Therefore, the framework may contribute to improved preparedness, better allocation of resources, and more resilient religious-tourism management in Iraq.

The study also has limitations. The current model is formulated as a general decision-support framework and requires detailed empirical calibration using real daily data on pilgrim counts, route density, service availability, incident reports, transport movement, weather conditions, and visitor satisfaction surveys. Future research may extend the model by incorporating spatial route networks, real-time mobile-density data, multi-city pilgrimage flows, stochastic uncertainty, and machine-learning-assisted parameter estimation. Such extensions would strengthen the predictive capacity of the (q, τ) -management framework and support real-time planning for Arbaeen and other large-scale religious tourism events.

Conflict of Interest

The author declares that there is no known competing financial interests or personal relationships that could have influenced the work reported in this paper.

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