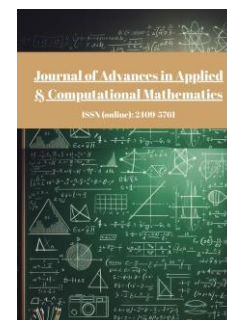




Published by Avanti Publishers

Journal of Advances in Applied & Computational Mathematics

ISSN (online): 2409-5761



Numerical Hypercyclicity of Composition Operators on Classical Function Spaces: Power Boundedness, Fixed Points, and Linear Dynamics

Hadi O. Alshammari¹, Otmane Benchiheb² and Marko Kostić^{3,*}

¹Mathematics Department, College of Science, Jouf University, Sakaka, Saudi Arabia

²Department of Mathematics, Faculty of Science, Chouaib Doukkali University, El Jadida, Morocco

³Faculty of Technical Sciences, University of Novi Sad, Novi Sad, Serbia

ARTICLE INFO

Article Type: Research Article

Academic Editor: Youssef Raffoul

Keywords:

Composition operators
Numerical hypercyclicity
Power bounded operators
Hardy and Bergman spaces
Hypercyclicity and linear dynamics

Timeline:

Received: March 02, 2026

Accepted: April 14, 2026

Published: April 24, 2026

Citation: Alshammari HO, Benchiheb O, Kostić M. Numerical hypercyclicity of composition operators on classical function spaces: Power boundedness, fixed points, and linear dynamics. J Adv Appl Computat Math. 2026; 13(1): 54-73.

DOI: <https://doi.org/10.15377/2409-5761.2026.13.4>

ABSTRACT

This paper investigates numerical hypercyclicity for unweighted composition operators on several classical Banach and Hilbert spaces of functions. The aim is to compare this scalar orbit-density property with ordinary hypercyclicity and to determine when the two notions coincide. We consider composition operators on $C([0,1])$, $H^\infty(D)$, the disc algebra $A(D)$, the Hardy space $H^2(D)$, the Bergman space $A^2(D)$, the Fock space $F^2(C)$, and the Dirichlet space D . The analysis combines power-boundedness arguments with known hypercyclicity criteria for composition operators. We show that power boundedness rules out both hypercyclicity and numerical hypercyclicity on $C([0,1])$, $H^\infty(D)$, $A(D)$, and for bounded composition operators on $F^2(C)$. In contrast, for automorphism-induced composition operators on $H^2(D)$ and $A^2(D)$, numerical hypercyclicity is equivalent to ordinary hypercyclicity, and hence to the absence of fixed points of the symbol in D . For the Dirichlet space, we obtain partial results: elliptic automorphisms are shown not to induce numerically hypercyclic composition operators, while the remaining automorphic cases are reduced to an explicit scalar-orbit problem. The novelty of the paper lies in providing a unified and space-dependent comparison between numerical hypercyclicity and hypercyclicity for composition operators across these classical settings.

2020 Mathematics Subject Classification: 47A16, 47B33, 30H10, 47B38.

*Corresponding Author

Email: marco.s@verat.net

Tel: +(38) 10641289790

1. Introduction

Linear dynamics studies the asymptotic behaviour of iterates of continuous linear operators on topological vector spaces. A central notion in this theory is hypercyclicity: an operator is hypercyclic if at least one of its orbits is dense in the underlying space. The first sources of this phenomenon go back to Birkhoff's universality theorem for translation operators on entire functions and to MacLane's corresponding result for the differentiation operator [1, 2]. In the Banach-space setting, Rolewicz gave a fundamental example by showing that suitable scalar multiples of the backward shift may be hypercyclic [3].

The modern development of the subject was shaped by several complementary directions. Kitai's thesis introduced influential criteria and invariant-set methods for linear dynamics [4]. Gethner and Shapiro studied universal vectors for operators acting on spaces of holomorphic functions, while Godefroy and Shapiro developed the theory of operators with dense invariant cyclic vector manifolds [5, 6]. Ansari clarified the relation between hypercyclic and cyclic vectors, and Salas gave fundamental examples and criteria for hypercyclic weighted shifts [7, 8]. Hereditary hypercyclicity was investigated by Bès and Peris, and rotations of hypercyclic and supercyclic operators were studied by León-Saavedra and Müller [9, 10]. General accounts and systematic developments of linear dynamics and linear chaos can be found in the monographs of Bayart–Matheron, Grosse-Erdmann–Peris, and Kostić [11-13].

Several refinements of orbit-density have also played an important role in the subject. Peris proved that multihypercyclic operators are hypercyclic, while Bayart and Grivaux introduced frequent hypercyclicity as a quantitative strengthening of hypercyclicity [14, 15]. Bonilla and Grosse-Erdmann further developed frequently hypercyclic vectors and operators [16]. Recurrence properties of hypercyclic operators were studied by Bès, Menet, Peris and Puig, and frequent recurrence was later investigated by Bonilla, Grosse-Erdmann, López-Martínez and Peris [17, 18]. The relation between somewhere dense orbits and dense orbits was clarified by Bourdon and Feldman, while Costakis and Parissis connected frequent hypercyclicity with multiple recurrence [19, 20]. Chaotic and distributionally chaotic behaviour has been studied in several forms, including the works of Bermúdez–Bonilla–Martínez–Giménez–Peris and Herrero [21, 22]. Further developments concerning weighted shifts include the works of Feldman, Martínez–Giménez–Peris, and Salas [23-25].

More recent work has considered cyclicity and recurrence-type properties for sets of operators and related structures. Amouch and Benchiheb studied cyclic sets of operators, linear dynamics for sets of operators, diskcyclicity, several versions of supercyclicity, super-recurrence, and codiskcyclicity [26-31]. Super-rigidity and weak recurrence were investigated in [32, 33]. Subspace-super recurrence and disjoint topological super-recurrence were considered in [34, 35]. In the setting of composition operators, Karim, Benchiheb and Amouch studied recurrence of multiples of composition operators on weighted Dirichlet spaces, while Benchiheb, Karim and Amouch considered compositional dynamics on the Hardy space [36, 37].

Composition operators form a natural bridge between operator theory and function theory. If ϕ is a self-map of a function domain, the operator $C_\phi f = f \circ \phi$ reflects both the geometry of the symbol ϕ and the structure of the function space on which it acts. The general theory of composition operators on analytic function spaces is treated in the monographs of Shapiro and Cowen–MacCluer [38, 39]. In the context of linear dynamics, Bernal-González and Montes-Rodríguez studied universal functions for composition operators, and Bourdon and Shapiro developed cyclic phenomena for such operators [40, 41]. Bayart examined common hypercyclic vectors for composition operators, while Grosse-Erdmann and Mortini studied universal functions associated with non-automorphic symbols [42, 43]. The cyclic behaviour of linear-fractional composition operators was studied by Gallardo-Gutiérrez and Montes-Rodríguez [44, 45]. On the Fock space, bounded composition operators have a very restrictive structure, as shown in the works of Carswell–MacCluer–Schuster and Guo–Izuchi [46, 47]. Recurrent composition operators on classical spaces were later studied by Costakis, Manoussos and Parissis [48].

The present paper focuses on the scalar version of orbit-density behaviour known as *numerical hypercyclicity*. Instead of asking whether a vector orbit is dense, one studies scalar sequences obtained by evaluating the orbit

through a norming functional. Thus, for a bounded operator T on a complex normed space X , one considers numerical orbits of the form

$$\{x^*(T^n x) : n \geq 0\}. \quad (1.1)$$

Kim, Peris and Song introduced this notion and showed that it leads to phenomena different from classical hypercyclicity [49]. Shkarin subsequently developed structural and finite-dimensional aspects of numerical hypercyclicity [50].

The mathematical motivation behind numerical hypercyclicity is to examine orbit-density through scalar observations rather than only through the full vector orbit. In many operator-theoretic situations, it is natural to ask not only whether the sequence $\{T^n x : n \geq 0\}$ fills the ambient space, but also how this orbit is detected by continuous linear functionals. Numerical hypercyclicity captures precisely this viewpoint by asking whether a scalar sequence of the form

$$x^*(T^n x), \quad n \geq 0, \quad (1.2)$$

can be dense in the complex plane for a norming pair (x, x^*) . In this sense, it fills a gap between classical vector-orbit dynamics and purely scalar operator-theoretic information: it may reveal rich scalar dynamics even when dense vector orbits are impossible. This distinction explains, for instance, why numerical hypercyclicity can occur in finite-dimensional complex spaces, whereas ordinary hypercyclicity cannot.

Our aim is to study numerical hypercyclicity for unweighted composition operators on several classical Banach and Hilbert spaces and to compare the resulting picture with the known hypercyclicity theory. More precisely, we consider operators C_ϕ acting on

$$C([0,1]), \quad H^\infty(D), \quad A(D), \quad H^2(D), \quad A^2(D), \quad F^2(C), \quad (1.3)$$

and we also discuss the Dirichlet space D . The guiding problem is to understand whether numerical hypercyclicity and hypercyclicity coincide for composition operators, and, when they do not, to identify the mechanism responsible for the difference.

The results reveal two distinct regimes. In the first one, the dynamics is rigid. On $C([0,1])$, $H^\infty(D)$, and $A(D)$, every composition operator considered here is power bounded, which rules out both hypercyclicity and numerical hypercyclicity. The same obstruction applies to bounded composition operators on the classical Fock space. Hence, in these spaces, passing from vector orbits to numerical orbits does not create new dense-orbit behaviour.

The second regime appears on the Hardy and Bergman spaces. For symbols that are automorphisms of the unit disc, we prove that numerical hypercyclicity of C_ϕ on $H^2(D)$ and on $A^2(D)$ is equivalent to ordinary hypercyclicity. Consequently, in the automorphic case, both properties are characterized by the absence of a fixed point of ϕ in D . We also examine the linear-fractional setting. On $H^2(D)$, the only linear-fractional symbols left unresolved from the numerical point of view are the parabolic non-automorphisms. On $A^2(D)$, our results reduce the corresponding problem to essentially the same remaining family.

The Dirichlet space behaves differently. In the automorphic case, the classical hypercyclicity theory is already negative. Numerically, we prove that elliptic automorphisms cannot induce numerically hypercyclic composition operators, and for the remaining automorphisms we reduce the question to an explicit scalar-orbit problem. Thus, unlike the Hardy and Bergman spaces, the Dirichlet space leads to a more delicate numerical problem rather than to a direct equivalence with hypercyclicity.

We deliberately keep the scope within Banach and Hilbert spaces, where the standard definition of numerical hypercyclicity applies without further modification. We do not introduce here a Fréchet-space version for spaces such as $H(C)$ or $H(D)$. Although composition operators on such spaces form an important part of classical hypercyclicity theory, the appropriate numerical formulation in the Fréchet setting requires separate treatment.

The paper is arranged as follows. First, we recall the basic definitions and record a simple obstruction based on power boundedness. We then treat the rigid negative cases in one section. The next two sections deal with the Hardy and Bergman spaces, where the automorphism case gives a positive comparison between hypercyclicity and numerical hypercyclicity, together with refinements for linear-fractional symbols. We then discuss the Dirichlet space, proving the elliptic obstruction and reducing the remaining automorphic cases to a scalar sequence. The final section summarizes the conclusions and formulates open problems concerning parabolic non-automorphisms, the Dirichlet space, and weighted composition operators.

Throughout the paper, D denotes the open unit disc and $Aut(D)$ denotes the group of automorphisms of D . If ϕ is a self-map of the underlying domain, then $\phi^{[n]}$ denotes its n -th iterate and C_ϕ denotes the composition operator $C_\phi f = f \circ \phi$. For a complex Banach space X , $L(X)$ denotes the algebra of bounded linear operators on X , and

$$\Pi(X) = \{(x, x^*) \in X \times X^*: \|x\| = \|x^*\| = 1, x^*(x) = 1\}$$

denotes the set of norming pairs. The abbreviations HC and NHC stand for hypercyclicity and numerical hypercyclicity, respectively.

2. Power-bounded Composition Operators: Rigid Negative Cases

This section is devoted to a simple obstruction which will be used repeatedly. On several classical Banach spaces, the composition operators considered below have uniformly bounded iterates. This boundedness prevents the existence of dense vector orbits and, at the same time, prevents the corresponding scalar numerical orbits from being dense in the complex plane. Thus all the negative results in this section follow from the same elementary principle.

Proposition 2.1 Let X be a complex Banach space and let $T \in L(X)$, where $L(X)$ denotes the algebra of all continuous linear operators on X . If

$$\sup_{n \geq 0} \|T^n\| < \infty, \tag{2.1}$$

then T is neither numerically hypercyclic nor hypercyclic.

Proof. Put

$$M = \sup_{n \geq 0} \|T^n\| < \infty, \tag{2.2}$$

For any $(x, f) \in \Pi(X)$ and every $n \geq 0$, we have

$$|f(T^n x)| \leq \|f\| \|T^n x\| \leq \|T^n\| \|x\| \leq M. \tag{2.3}$$

Hence every numerical orbit associated with T is contained in the closed disc of radius M in \mathbb{C} . It cannot therefore be dense in \mathbb{C} , and T is not numerically hypercyclic.

Similarly, for every $x \in X$,

$$\|T^n x\| \leq M \|x\| \quad (n \geq 0). \tag{2.4}$$

Thus each orbit of T is bounded in X . Since a bounded subset of an infinite-dimensional Banach space cannot be dense in the whole space, no orbit of T is dense. Consequently, T is not hypercyclic.

Remark 2.2 The preceding proposition gives only one obstruction to numerical hypercyclicity; it should not be understood as a characterization. There are also non-power-bounded operators which fail to be numerically hypercyclic for reasons unrelated to uniform boundedness of the iterates. For example, on \mathbb{C}^2 with its usual Hilbert norm, let

$$T = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}. \tag{2.5}$$

Then T is not power bounded. However, for every unit vector $x = (x_1, x_2) \in C^2$, the corresponding Hilbert-space numerical orbit satisfies

$$\langle T^n x, x \rangle = 2^n |x_1|^2 + 3^n |x_2|^2 \in [0, \infty) \quad (n \geq 0). \tag{2.6}$$

Thus no such numerical orbit can be dense in C . This illustrates that numerical hypercyclicity may also be excluded by the geometry of scalar numerical orbits, even when the operator is not power bounded. In the present paper, the non-power-bounded composition-operator cases that remain most delicate are the parabolic non-automorphic linear-fractional symbols; these are isolated below as open problems because their resolution requires scalar orbit estimates beyond the power-bounded argument.

Composition operators on $C([0,1])$.

Let ϕ be a continuous self-map of $[0,1]$. We write

$$C_\phi f = f \circ \phi, \quad f \in C([0,1]), \tag{2.7}$$

and we use the norm

$$\|f\|_\infty = \max_{x \in [0,1]} |f(x)|. \tag{2.8}$$

Proposition 2.3 For every continuous self-map ϕ of $[0,1]$, the operator C_ϕ is power bounded on $C([0,1])$. Consequently, it is neither hypercyclic nor numerically hypercyclic.

Proof. Since $\phi([0,1]) \subseteq [0,1]$, every $f \in C([0,1])$ satisfies

$$\|C_\phi f\|_\infty = \max_{x \in [0,1]} |f(\phi(x))| \leq \max_{y \in [0,1]} |f(y)| = \|f\|_\infty. \tag{2.9}$$

Thus C_ϕ is contractive. Since

$$C_\phi^n = C_{\phi^{[n]}} \quad (n \geq 0), \tag{2.10}$$

and since each $\phi^{[n]}$ still maps $[0,1]$ into itself, the same estimate gives

$$\|C_\phi^n f\|_\infty \leq \|f\|_\infty \quad (f \in C([0,1]), n \geq 0). \tag{2.11}$$

Hence

$$\sup_{n \geq 0} \|C_\phi^n\| \leq 1. \tag{2.12}$$

The result follows from Proposition 2.1.

Composition operators on $H^\infty(D)$.

Let ϕ be a holomorphic self-map of D . On the space

$$H^\infty(D) = \{f \in H(D) : \|f\|_\infty < \infty\}, \quad \|f\|_\infty = \sup_{z \in D} |f(z)|, \tag{2.13}$$

we consider the composition operator

$$C_\phi f = f \circ \phi. \tag{2.14}$$

Proposition 2.4 For every holomorphic self-map ϕ of D , the operator C_ϕ satisfies

$$\|C_\phi\| = 1 \quad (2.15)$$

on $H^\infty(D)$. In particular, C_ϕ is power bounded and therefore cannot be hypercyclic or numerically hypercyclic.

Proof. For $f \in H^\infty(D)$, the range condition $\phi(D) \subseteq D$ gives

$$\|C_\phi f\|_\infty = \sup_{z \in D} |f(\phi(z))| \leq \sup_{w \in D} |f(w)| = \|f\|_\infty. \quad (2.16)$$

Hence $PC_\phi P \leq 1$. Since the constant function $\mathbf{1}$ is left unchanged by C_ϕ , we also have

$$\|C_\phi\| \geq 1. \quad (2.17)$$

Thus $PC_\phi P = 1$.

Moreover,

$$C_\phi^n = C_{\phi^{[n]}} \quad (n \geq 0), \quad (2.18)$$

and every iterate $\phi^{[n]}$ is again a holomorphic self-map of D . Repeating the preceding estimate for $\phi^{[n]}$, we get

$$\|C_\phi^n\| = 1 \quad (n \geq 0). \quad (2.19)$$

Thus C_ϕ is power bounded, and Proposition 2.1 yields the conclusion.

Composition operators on the disc algebra $A(D)$.

Let $\phi \in A(D)$ satisfy

$$\phi(\bar{D}) \subseteq \bar{D}. \quad (2.20)$$

On the disc algebra

$$A(D) = H(D) \cap C(\bar{D}), \quad \|f\|_\infty = \max_{z \in \bar{D}} |f(z)|, \quad (2.21)$$

the induced composition operator is

$$C_\phi f = f \circ \phi. \quad (2.22)$$

Proposition 2.5 If ϕ satisfies (2.20), then

$$\|C_\phi\| = 1 \quad (2.23)$$

on $A(D)$. Hence C_ϕ is power bounded and cannot be hypercyclic or numerically hypercyclic.

Proof. For $f \in A(D)$, condition (2.20) gives

$$\|C_\phi f\|_\infty = \max_{z \in \bar{D}} |f(\phi(z))| \leq \max_{w \in \bar{D}} |f(w)| = \|f\|_\infty. \quad (2.24)$$

Therefore $PC_\phi P \leq 1$. Since constants are fixed, we also have

$$\|C_\phi\| \geq 1. \quad (2.25)$$

Thus $PC_\phi P = 1$.

Furthermore,

$$C_\phi^n = C_{\phi^{[n]}} \quad (n \geq 0), \tag{2.26}$$

and each $\phi^{[n]}$ still maps \bar{D} into itself. Repeating (2.24) for $\phi^{[n]}$, we get

$$\|C_\phi^n\| = 1 \quad (n \geq 0). \tag{2.27}$$

The conclusion follows from Proposition 2.1.

Remark 2.6 The condition $\phi(\bar{D}) \subseteq \bar{D}$ does not force the symbol to stay away from the boundary. Boundary contact, including the case where the Denjoy–Wolff point lies on ∂D , is still allowed whenever the symbol belongs to $A(D)$. However, on the disc algebra, this boundary behaviour is dominated by the uniform norm:

$$\|C_\phi f\|_\infty \leq \|f\|_\infty. \tag{2.28}$$

Thus the boundary dynamics that may be decisive in Hardy or Bergman spaces does not produce hypercyclicity or numerical hypercyclicity for unweighted composition operators on $A(D)$.

Composition operators on the Fock space.

We now consider

$$F^2(\mathbb{C}) = \left\{ f \in H(\mathbb{C}) : \|f\|_{F^2}^2 = \frac{1}{2\pi} \int_{\mathbb{C}} |f(z)|^2 e^{-\frac{|z|^2}{2}} dA(z) < \infty \right\}. \tag{2.29}$$

For an entire symbol ϕ , we write

$$C_\phi f = f \circ \phi, \tag{2.30}$$

whenever this defines a bounded operator on $F^2(\mathbb{C})$.

Proposition 2.7 Let $\phi: \mathbb{C} \rightarrow \mathbb{C}$ be entire and suppose that C_ϕ is bounded on $F^2(\mathbb{C})$. Then

$$\phi(z) = az + b, \tag{2.31}$$

where either $|a| < 1$, or $|a| = 1$ and $b = 0$. In both cases C_ϕ is power bounded. Hence it is neither hypercyclic nor numerically hypercyclic.

Proof. The boundedness characterization on the Fock space gives the affine form

$$\phi(z) = az + b, \tag{2.32}$$

with either $|a| < 1$, or $|a| = 1$ and $b = 0$.

If $|a| = 1$ and $b = 0$, then $\phi(z) = az$. Since the change of variables $w = az$ preserves both dA and $|z|$, we obtain

$$\begin{aligned} \|C_\phi f\|_{F^2}^2 &= \frac{1}{2\pi} \int_{\mathbb{C}} |f(az)|^2 e^{-\frac{|z|^2}{2}} dA(z) \\ &= \frac{1}{2\pi} \int_{\mathbb{C}} |f(w)|^2 e^{-\frac{|w|^2}{2}} dA(w) \\ &= \|f\|_{F^2}^2. \end{aligned} \tag{2.33}$$

Thus C_ϕ is unitary and, in particular, power bounded.

Assume now that $|a| < 1$. The iterates of ϕ are

$$\phi^{[n]}(z) = a^n z + b_n, \quad b_n = b \sum_{j=0}^{n-1} a^j = \frac{b(1-a^n)}{1-a}. \quad (2.34)$$

Hence

$$|b_n| \leq \frac{2|b|}{|1-a|} \quad (n \geq 1). \quad (2.35)$$

For the normalization used in (2.29), the reproducing kernel is

$$K_w(z) = \exp\left(\frac{z\bar{w}}{2}\right). \quad (2.36)$$

Moreover, if

$$\psi(z) = \alpha z + \beta, \quad |\alpha| < 1, \quad (2.37)$$

then the norm of the induced composition operator is

$$\|C_\psi\| = \exp\left(\frac{|\beta|^2}{4(1-|\alpha|^2)}\right). \quad (2.38)$$

Applying this formula with

$$\psi = \phi^{[n]}, \quad \alpha = a^n, \quad \beta = b_n, \quad (2.39)$$

gives

$$\|C_\phi^n\| = \|C_{\phi^{[n]}}\| = \exp\left(\frac{|b_n|^2}{4(1-|a|^{2n})}\right). \quad (2.40)$$

Since

$$1 - |a|^{2n} \geq 1 - |a|^2 \quad (n \geq 1), \quad (2.41)$$

we infer from (2.35) that

$$\|C_\phi^n\| \leq \exp\left(\frac{|b|^2}{|1-a|^2(1-|a|^2)}\right) \quad (n \geq 1). \quad (2.42)$$

Therefore

$$\sup_{n \geq 0} \|C_\phi^n\| < \infty. \quad (2.43)$$

The result now follows from Proposition 2.1.

3. Composition Operators on $H^2(D)$

We now move to the Hardy space

$$H^2(D) = \{f(z) = \sum_{n=0}^{\infty} a_n z^n : \sum_{n=0}^{\infty} |a_n|^2 < \infty\}, \quad \|f\|_{H^2}^2 = \sum_{n=0}^{\infty} |a_n|^2. \quad (3.1)$$

Unlike the spaces treated in the previous section, $H^2(D)$ permits nontrivial composition-operator dynamics. In particular, when the symbol is an automorphism of the disc, the classical hypercyclic behaviour is governed by whether the symbol has an interior fixed point. The purpose of this section is to show that the same dichotomy also governs numerical hypercyclicity in the automorphic case.

If $\phi: D \rightarrow D$ is analytic, then

$$C_\phi: H^2(D) \rightarrow H^2(D), \quad C_\phi f = f \circ \phi, \tag{3.2}$$

is bounded.

Theorem 3.1 (Bourdon--Shapiro; see also [Theorem 1.1] Bayart 2004) Let $\phi \in \text{Aut}(D)$. Then C_ϕ is hypercyclic on $H^2(D)$ if and only if ϕ has no fixed point in D .

The next proposition excludes numerical hypercyclicity whenever the symbol has an interior fixed point.

Proposition 3.2 Let $\phi: D \rightarrow D$ be analytic and suppose that $\phi(a) = a$ for some $a \in D$. Then C_ϕ is power bounded on $H^2(D)$. Consequently, C_ϕ is neither hypercyclic nor numerically hypercyclic.

Proof. We distinguish the elliptic automorphic case from the remaining case.

Assume first that ϕ is not an elliptic automorphism. Since ϕ has the fixed point $a \in D$, the Denjoy–Wolff theorem implies that its iterates converge locally uniformly to a . In particular,

$$\phi^{[n]}(0) \rightarrow a. \tag{3.3}$$

For every analytic self-map ψ of D , the standard Littlewood-type estimate on $H^2(D)$ gives

$$\|C_\psi\| \leq \left(\frac{1+|\psi(0)|}{1-|\psi(0)|} \right)^{1/2}. \tag{3.4}$$

Using this estimate with $\psi = \phi^{[n]}$, we obtain

$$\|C_\phi^n\| = \|C_{\phi^{[n]}}\| \leq \left(\frac{1+|\phi^{[n]}(0)|}{1-|\phi^{[n]}(0)|} \right)^{1/2} \quad (n \geq 0). \tag{3.5}$$

Because the sequence $\phi^{[n]}(0)$ converges to a point strictly inside D , the right-hand side of (3.5) remains bounded. Hence C_ϕ is power bounded.

It remains to consider elliptic automorphisms. In this case ϕ is conjugate to a rotation. More precisely, if

$$\alpha_a(z) = \frac{a-z}{1-\bar{a}z}, \tag{3.6}$$

then there exists $\lambda \in \mathbb{C}$, $|\lambda| = 1$, such that

$$\alpha_a \circ \phi \circ \alpha_a(z) = \lambda z. \tag{3.7}$$

With $r_\lambda(z) = \lambda z$, this gives

$$\phi = \alpha_a \circ r_\lambda \circ \alpha_a, \quad C_\phi = C_{\alpha_a} C_{r_\lambda} C_{\alpha_a}. \tag{3.8}$$

Consequently,

$$C_\phi^n = C_{\alpha_a} C_{r_\lambda}^n C_{\alpha_a} \quad (n \geq 0). \tag{3.9}$$

The rotation operator C_{r_λ} preserves the Hardy norm. Indeed, for $f(z) = \sum_{k=0}^\infty a_k z^k$,

$$(C_{r_\lambda} f)(z) = f(\lambda z) = \sum_{k=0}^\infty a_k \lambda^k z^k, \tag{3.10}$$

and hence

$$\|C_{r_\lambda} f\|_{H^2}^2 = \sum_{k=0}^\infty |a_k \lambda^k|^2 = \sum_{k=0}^\infty |a_k|^2 = \|f\|_{H^2}^2. \tag{3.11}$$

Thus $PC_{r_\lambda}^n P = 1$ for all $n \geq 0$. From (3.9), we get

$$\|C_\phi^n\| \leq \|C_{\alpha_\alpha}\|^2 \quad (n \geq 0), \quad (3.12)$$

so C_ϕ is power bounded also in the elliptic automorphic case.

The conclusion follows from Proposition 2.1.

We now obtain the full automorphic comparison on the Hardy space.

Theorem 3.3 Let $\phi \in \text{Aut}(D)$. The following assertions are equivalent:

1. C_ϕ is hypercyclic on $H^2(D)$;
2. C_ϕ is numerically hypercyclic on $H^2(D)$;
3. ϕ has no fixed point in D .

Proof. Assume that C_ϕ is hypercyclic and choose a hypercyclic vector x . After normalization, we may suppose that $PxP = 1$. By Hahn–Banach, there exists $f \in (H^2(D))^*$ such that

$$\|f\| = 1, \quad f(x) = 1. \quad (3.13)$$

Thus $(x, f) \in \Pi(H^2(D))$.

Let $\mu \in \mathbb{C}$ and let $\varepsilon > 0$. Since the orbit of x is dense, one can choose $n \geq 0$ with

$$\|C_\phi^n x - \mu x\| < \varepsilon. \quad (3.14)$$

Applying f gives

$$\begin{aligned} |f(C_\phi^n x) - \mu| &= |f(C_\phi^n x - \mu x)| \\ &\leq \|f\| \|C_\phi^n x - \mu x\| \\ &< \varepsilon. \end{aligned} \quad (3.15)$$

Hence

$$\{f(C_\phi^n x): n \geq 0\} \quad (3.16)$$

is dense in \mathbb{C} , so C_ϕ is numerically hypercyclic.

Conversely, suppose that C_ϕ is numerically hypercyclic. If ϕ had a fixed point in D , then Proposition 3.2 would contradict numerical hypercyclicity. Therefore ϕ has no fixed point in D .

Finally, Theorem 3.1 gives the equivalence between the absence of an interior fixed point and hypercyclicity of C_ϕ .

Thus, for automorphic symbols on $H^2(D)$,

$$\begin{aligned} C_\phi \text{ is hypercyclic} &\Leftrightarrow C_\phi \text{ is numerically hypercyclic} \\ &\Leftrightarrow \phi \text{ has no fixed point in } D. \end{aligned} \quad (3.17)$$

Equivalently, among automorphisms of the disc, the numerically hypercyclic composition operators are precisely those associated with parabolic or hyperbolic automorphisms.

We now record the corresponding reduction for linear-fractional symbols.

Proposition 3.4 Let ϕ be a linear-fractional self-map of D . Then:

1. if ϕ has a fixed point in D , then C_ϕ is neither hypercyclic nor numerically hypercyclic on $H^2(D)$;
2. if ϕ has no fixed point in D and is not parabolic non-automorphic, then C_ϕ is hypercyclic and therefore numerically hypercyclic on $H^2(D)$.

Hence the only linear-fractional case not settled here by our numerical arguments is the parabolic non-automorphic case.

Proof. The first item follows directly from Proposition 3.2.

For the second item, we use the Bourdon–Shapiro classification. In the linear-fractional class, a non-elliptic symbol without fixed points in D induces a hypercyclic composition operator on $H^2(D)$, apart from the parabolic non-automorphic family; see [41, Theorem 2.2]. Therefore C_ϕ is hypercyclic under the assumptions in item 2. The Hahn–Banach argument used in the proof of Theorem 3.3 then gives numerical hypercyclicity.

Thus the only unresolved linear-fractional family in the Hardy-space case is formed by parabolic non-automorphisms. Bourdon and Shapiro showed that, for such symbols, the cluster points of ordinary C_ϕ -orbits are only constant functions [41, Theorem 2.4(b)]. Whether this severe degeneration of vector orbits also rules out dense numerical orbits remains open.

Remark 3.5 The parabolic non-automorphism case is not reached by the two main tools used above. The power-bounded obstruction does not settle it, and the implication from hypercyclicity to numerical hypercyclicity is unavailable because these symbols are not hypercyclic in the known linear-fractional classification. The obstacle is therefore not simply spectral. What is missing is a sufficiently fine description of scalar numerical orbits

$$x^*(C_\phi^n x), \quad n \geq 0, \tag{3.18}$$

for arbitrary norming pairs (x, x^*) . A solution of this case would require estimates designed specifically for the scalar behaviour generated by parabolic non-automorphic symbols.

4. Composition Operators on the Bergman Space $A^2(D)$

We now consider the Bergman space

$$A^2(D) = \{f \in H(D) : \int_D |f(z)|^2 dA(z) < \infty\}, \quad \|f\|_{A^2}^2 = \int_D |f(z)|^2 dA(z), \tag{4.1}$$

where dA denotes normalized area measure on D . Although this space is close in spirit to the Hardy space, its norm is area-based rather than boundary-based. Nevertheless, for automorphic composition operators, the same fixed-point alternative governs the comparison between hypercyclicity and numerical hypercyclicity.

If $\phi: D \rightarrow D$ is analytic, then

$$C_\phi: A^2(D) \rightarrow A^2(D), \quad C_\phi f = f \circ \phi, \tag{4.2}$$

is bounded.

We shall use the following known automorphic characterization.

Theorem 4.1 (Gallardo--Gutiàrez and Montes--Rodr_uez; cf.)

Let $\phi \in \text{Aut}(D)$. Then C_ϕ is hypercyclic on $A^2(D)$ if and only if ϕ has no fixed point in D .

We first remove the elliptic automorphic case from the numerical picture.

Proposition 4.2 Let $\phi \in \text{Aut}(D)$ have a fixed point in D . Then C_ϕ is power bounded on $A^2(D)$. In particular, it is neither hypercyclic nor numerically hypercyclic on $A^2(D)$.

Proof. An automorphism of D with an interior fixed point is elliptic. Let $a \in D$ be such a fixed point. Consider

$$\alpha_a(z) = \frac{a-z}{1-\bar{a}z}. \quad (4.3)$$

Then α_a interchanges 0 and a , and ϕ is conjugate to a rotation. Thus there exists $\lambda \in \mathbb{C}$, $|\lambda| = 1$, such that

$$\alpha_a \circ \phi \circ \alpha_a(z) = \lambda z. \quad (4.4)$$

Equivalently, with $r_\lambda(z) = \lambda z$,

$$\phi = \alpha_a \circ r_\lambda \circ \alpha_a. \quad (4.5)$$

Consequently,

$$C_\phi = C_{\alpha_a} C_{r_\lambda} C_{\alpha_a}, \quad (4.6)$$

and hence, for all $n \geq 0$,

$$C_\phi^n = C_{\alpha_a} C_{r_\lambda}^n C_{\alpha_a}. \quad (4.7)$$

The rotation part acts isometrically on $A^2(D)$. Indeed, by the change of variables $w = \lambda z$, we get

$$\begin{aligned} \|C_{r_\lambda} f\|_{A^2}^2 &= \int_D |f(\lambda z)|^2 dA(z) \\ &= \int_D |f(w)|^2 dA(w) \\ &= \|f\|_{A^2}^2. \end{aligned} \quad (4.8)$$

Therefore

$$\|C_{r_\lambda}^n\| = 1 \quad (n \geq 0). \quad (4.9)$$

Since C_{α_a} is bounded on $A^2(D)$, the preceding factorization gives

$$\|C_\phi^n\| \leq \|C_{\alpha_a}\|^2 \quad (n \geq 0). \quad (4.10)$$

Thus C_ϕ is power bounded, and Proposition 2.1 completes the proof.

We can now state the Bergman-space analogue of the Hardy-space comparison.

Theorem 4.3 Let $\phi \in \text{Aut}(D)$. The following assertions are equivalent:

1. C_ϕ is hypercyclic on $A^2(D)$;
2. C_ϕ is numerically hypercyclic on $A^2(D)$;
3. ϕ has no fixed point in D .

Proof. The implication from hypercyclicity to numerical hypercyclicity follows by the same Hahn–Banach argument used in the Hardy-space case. For completeness, we recall the short argument. Let f be a hypercyclic vector and normalize it so that $PfP = 1$. Choose $\Lambda \in (A^2(D))^*$ such that

$$\|\Lambda\| = 1, \quad \Lambda(f) = 1. \quad (4.11)$$

Given $\mu \in C$ and $\varepsilon > 0$, the density of the orbit of f gives some $n \geq 0$ satisfying

$$\|C_\phi^n f - \mu f\| < \varepsilon. \tag{4.12}$$

Hence

$$\begin{aligned} |\Lambda(C_\phi^n f) - \mu| &= |\Lambda(C_\phi^n f - \mu f)| \\ &\leq \|\Lambda\| \|C_\phi^n f - \mu f\| \\ &< \varepsilon. \end{aligned} \tag{4.13}$$

It follows that

$$\{\Lambda(C_\phi^n f): n \geq 0\} \tag{4.14}$$

is dense in C .

Conversely, if C_ϕ were numerically hypercyclic while ϕ had an interior fixed point, Proposition 4.2 would give a contradiction. Hence ϕ has no fixed point in D . Theorem 4.1 then yields hypercyclicity.

Consequently, for automorphic symbols on the Bergman space,

$$\begin{aligned} C_\phi \text{ is hypercyclic} &\Leftrightarrow C_\phi \text{ is numerically hypercyclic} \\ &\Leftrightarrow \phi \text{ has no fixed point in } D. \end{aligned} \tag{4.15}$$

We finally record the corresponding linear-fractional reduction.

Proposition 4.4 Let ϕ be a linear-fractional self-map of D . Then:

1. if ϕ has a fixed point in D , then C_ϕ is not hypercyclic on $A^2(D)$. If ϕ is moreover an automorphism, then C_ϕ is neither hypercyclic nor numerically hypercyclic;
2. if ϕ has no fixed point in D and does not belong to the parabolic non-automorphic family, then C_ϕ is hypercyclic and therefore numerically hypercyclic on $A^2(D)$.

Thus, among linear-fractional symbols without interior fixed points, the only case left open here for numerical hypercyclicity is the parabolic non-automorphic case.

Proof. Recall that $A^2(D)$ corresponds to $S_{-1/2}$ in the weighted Dirichlet scale. The classification of Gallardo-Gutiérrez and Montes-Rodríguez for linear-fractional composition operators on S_ν shows that, for $\nu < 1/2$, the hypercyclic cases are precisely the hyperbolic automorphisms, the parabolic automorphisms, and the hyperbolic non-automorphisms. Hence a linear-fractional symbol with an interior fixed point cannot induce a hypercyclic operator on $A^2(D)$. If this symbol is also an automorphism, Proposition 4.2 gives the stronger numerical conclusion.

Now assume that ϕ has no fixed point in D and is not parabolic non-automorphic. The same classification gives hypercyclicity of C_ϕ on $A^2(D)$. The Hahn–Banach argument above then implies numerical hypercyclicity.

Therefore, in the Bergman linear-fractional setting, the only remaining no-fixed-point family is the parabolic non-automorphic one. The available classification excludes classical hypercyclicity for these symbols, but it does not by itself decide numerical hypercyclicity. Likewise, for non-automorphic symbols with an interior fixed point, we use only the known negative hypercyclicity statement and do not claim a complete numerical classification.

Remark 4.5 The automorphism case is completely settled by Theorem 4.3. Proposition 4.4 shows that the only unresolved linear-fractional case without an interior fixed point is the parabolic non-automorphic family. We do not claim a characterization for arbitrary analytic symbols, nor for all non-automorphic linear-fractional symbols with an interior fixed point.

5. Composition Operators on the Dirichlet Space

We finally discuss the Dirichlet space

$$D = \{f(z) = \sum_{n=0}^{\infty} a_n z^n : \sum_{n=0}^{\infty} (n+1) |a_n|^2 < \infty\}, \quad \|f\|_D^2 = \sum_{n=0}^{\infty} (n+1) |a_n|^2. \quad (5.1)$$

This is the weighted Dirichlet space $S_{1/2}$. Its composition-operator dynamics is more rigid than the Hardy and Bergman cases considered above. In particular, for automorphic symbols, classical hypercyclicity is absent. Our goal in this section is therefore more limited: we first exclude numerical hypercyclicity for elliptic automorphisms, and then we express the remaining automorphic numerical problem through a concrete scalar sequence.

For $\phi \in \text{Aut}(D)$, the induced operator

$$C_\phi: D \rightarrow D, \quad C_\phi f = f \circ \phi, \quad (5.2)$$

is bounded and invertible.

We shall use the following known result.

Theorem 5.1 (Gallardo--Gutiàrez and Montes--Rodr_uez; cf.)

Let $\phi \in \text{Aut}(D)$. Then C_ϕ is not hypercyclic on D .

We start with the elliptic automorphic case.

Proposition 5.2 If $\phi \in \text{Aut}(D)$ is elliptic, then C_ϕ is power bounded on D . Consequently, C_ϕ is not numerically hypercyclic on D .

Proof. Let $a \in D$ be the fixed point of ϕ . Consider the involutive automorphism

$$\alpha_a(z) = \frac{a-z}{1-\bar{a}z}. \quad (5.3)$$

It sends 0 to a and a to 0. Since ϕ is elliptic, there exists a unimodular number λ such that

$$\alpha_a \circ \phi \circ \alpha_a(z) = \lambda z. \quad (5.4)$$

Equivalently, with $r_\lambda(z) = \lambda z$, we have

$$\phi = \alpha_a \circ r_\lambda \circ \alpha_a. \quad (5.5)$$

Passing to composition operators gives

$$C_\phi = C_{\alpha_a} C_{r_\lambda} C_{\alpha_a}, \quad (5.6)$$

and therefore

$$C_\phi^n = C_{\alpha_a} C_{r_\lambda}^n C_{\alpha_a} \quad (n \geq 0). \quad (5.7)$$

It remains only to observe that rotations preserve the coefficient norm of D . Indeed, for

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, \quad (5.8)$$

we have

$$(C_{r_\lambda} f)(z) = f(\lambda z) = \sum_{k=0}^{\infty} a_k \lambda^k z^k. \quad (5.9)$$

Since $|\lambda| = 1$,

$$\begin{aligned} \|C_{r_\lambda} f\|_D^2 &= \sum_{k=0}^{\infty} (k+1) |a_k \lambda^k|^2 \\ &= \sum_{k=0}^{\infty} (k+1) |a_k|^2 \\ &= \|f\|_D^2. \end{aligned} \tag{5.10}$$

Thus $PC_{r_\lambda}^n P = 1$ for every $n \geq 0$. Since C_{α_a} is bounded on D , (5.7) yields

$$\|C_\phi^n\| \leq \|C_{\alpha_a}\|^2 \quad (n \geq 0). \tag{5.11}$$

Hence C_ϕ is power bounded, and Proposition 2.1 excludes numerical hypercyclicity.

We now separate the constant part from the functions vanishing at the origin. Set

$$D_0 = \{f \in D: f(0) = 0\}. \tag{5.12}$$

With respect to the coefficient inner product, one has the orthogonal decomposition

$$D = C \oplus D_0, \tag{5.13}$$

and each $f \in D$ decomposes as

$$f = f(0) + (f - f(0)). \tag{5.14}$$

Proposition 5.3 Let $\phi \in \text{Aut}(D)$. Relative to $D = C \oplus D_0$, the operator C_ϕ has the block form

$$C_\phi = \begin{pmatrix} 1 & \Lambda_\phi \\ 0 & U_\phi \end{pmatrix}, \tag{5.15}$$

where

$$\Lambda_\phi(u) = u(\phi(0)), \quad U_\phi u = u \circ \phi - u(\phi(0)). \tag{5.16}$$

Thus, for $c \in C$ and $u \in D_0$,

$$C_\phi(c + u) = (c + u(\phi(0))) \oplus U_\phi u. \tag{5.17}$$

Moreover,

$$U_\phi^n u = u \circ \phi^{[n]} - u(\phi^{[n]}(0)) \quad (n \geq 0), \tag{5.18}$$

and hence

$$C_\phi^n(c + u) = (c + u(\phi^{[n]}(0))) \oplus U_\phi^n u. \tag{5.19}$$

Proof. Let $u \in D_0$. Since point evaluations are continuous on D , the function

$$U_\phi u = u \circ \phi - u(\phi(0)) \tag{5.20}$$

belongs to D_0 . The boundedness of U_ϕ follows from the boundedness of C_ϕ and of the evaluation functional at $\phi(0)$.

For $c \in C$ and $u \in D_0$, we write

$$\begin{aligned} C_\phi(c + u) &= c + u \circ \phi \\ &= (c + u(\phi(0))) + (u \circ \phi - u(\phi(0))). \end{aligned} \tag{5.21}$$

The first summand is constant, while the second lies in D_0 . This proves the block representation, with

$$\Lambda_\phi(u) = u(\phi(0)). \quad (5.22)$$

It remains to justify the formula for U_ϕ^n . The case $n = 0$ is trivial. Assume that

$$U_\phi^n u = u \circ \phi^{[n]} - u(\phi^{[n]}(0)). \quad (5.23)$$

Then

$$\begin{aligned} U_\phi^{n+1} u &= U_\phi(U_\phi^n u) \\ &= (u \circ \phi^{[n]} - u(\phi^{[n]}(0))) \circ \phi \\ &\quad - (u \circ \phi^{[n]} - u(\phi^{[n]}(0)))(\phi(0)) \\ &= u \circ \phi^{[n+1]} - u(\phi^{[n+1]}(0)). \end{aligned} \quad (5.24)$$

This proves (5.18); formula (5.19) then follows from the block form.

The block representation gives a useful reduction of numerical orbits to a scalar sequence.

Corollary 5.4 Let $\phi \in \text{Aut}(D)$ and let $x = c + u \in D$, with $c \in \mathbb{C}$ and $u \in D_0$. Then

$$\langle C_\phi^n x, x \rangle_D = |c|^2 + \bar{c}u(\phi^{[n]}(0)) + \langle U_\phi^n u, u \rangle_D \quad (n \geq 0). \quad (5.25)$$

Consequently, C_ϕ is numerically hypercyclic on D if and only if, for some unit vector $x = c + u \in D$, the set

$$\{|c|^2 + \bar{c}u(\phi^{[n]}(0)) + \langle U_\phi^n u, u \rangle_D : n \geq 0\} \quad (5.26)$$

is dense in \mathbb{C} .

Proof. By Proposition 5.3,

$$C_\phi^n(c + u) = (c + u(\phi^{[n]}(0))) \oplus U_\phi^n u. \quad (5.27)$$

Using the orthogonality of

$$D = \mathbb{C} \oplus D_0, \quad (5.28)$$

we obtain

$$\begin{aligned} \langle C_\phi^n x, x \rangle_D &= (c + u(\phi^{[n]}(0)))\bar{c} + \langle U_\phi^n u, u \rangle_D \\ &= |c|^2 + \bar{c}u(\phi^{[n]}(0)) + \langle U_\phi^n u, u \rangle_D. \end{aligned} \quad (5.29)$$

It remains to relate this formula to the definition of numerical hypercyclicity. Let $(x, f) \in \Pi(D)$. Since D is a Hilbert space, there exists $y \in D$ such that

$$f(z) = \langle z, y \rangle_D, \quad \|y\|_D = 1. \quad (5.30)$$

The conditions defining $\Pi(D)$ give

$$\langle x, y \rangle_D = 1. \quad (5.31)$$

Equality in the Cauchy-Schwarz inequality then forces $y = x$. Hence the only numerical orbits to be considered are

$$\langle C_\phi^n x, x \rangle_D, \quad \|x\|_D = 1. \quad (5.32)$$

The desired criterion follows.

We close this section with the resulting automorphic picture.

Remark 5.5 Elliptic automorphisms do not give numerically hypercyclic composition operators on D . For parabolic and hyperbolic automorphisms, classical hypercyclicity is still ruled out by Theorem 5.1, but the numerical question reduces, by Corollary 5.4, to the density of the scalar sequence

$$|c|^2 + \bar{c}u(\phi^{[n]}(0)) + \langle U_\phi^n u, u \rangle_D. \tag{5.33}$$

Thus the Dirichlet space separates itself from the Hardy–Bergman pattern. In $H^2(D)$ and $A^2(D)$, non-elliptic automorphisms yield the same answer for hypercyclicity and numerical hypercyclicity. On D , however, classical hypercyclicity is absent for automorphisms, whereas the numerical question remains encoded in the scalar orbit above.

6. Concluding Remarks and Open Problems

The purpose of this work was to examine how numerical hypercyclicity behaves for composition operators on classical function spaces and to compare it with the usual notion of hypercyclicity. The results show that the answer depends strongly on the geometry of the underlying space and on the type of symbol which induces the operator.

A first group of spaces leads to a purely rigid situation. More precisely, on

$$C([0,1]), \quad H^\infty(D), \quad A(D), \quad F^2(C), \tag{6.1}$$

the relevant composition operators have uniformly bounded powers. This prevents the existence of dense vector orbits and also confines all numerical orbits to bounded subsets of the scalar field. Hence neither hypercyclicity nor numerical hypercyclicity can occur in these cases.

The Hardy and Bergman spaces give a different picture. For automorphic symbols on $H^2(D)$ and $A^2(D)$, the two notions studied in this paper agree completely. In both spaces, for $\phi \in \text{Aut}(D)$, one has

$$\begin{aligned} C_\phi \text{ is hypercyclic} &\Leftrightarrow C_\phi \text{ is numerically hypercyclic} \\ &\Leftrightarrow \phi \text{ has no fixed point in } D. \end{aligned} \tag{6.2}$$

Thus, within the automorphism class, passing from vector orbits to numerical orbits does not change the class of admissible symbols on the Hardy and Bergman spaces.

The Dirichlet space behaves in a more singular way. For automorphisms of D , classical hypercyclicity is always absent on D . We also excluded numerical hypercyclicity for elliptic automorphisms. For the remaining automorphisms, the problem was not reduced to a classical hypercyclicity criterion, but instead to a concrete scalar sequence. This places the Dirichlet space between the rigid negative cases and the Hardy–Bergman equivalence regime.

The main contribution of the paper is therefore threefold: it identifies power boundedness as a decisive obstruction on several standard spaces; it establishes an exact automorphic equivalence between hypercyclicity and numerical hypercyclicity on $H^2(D)$ and $A^2(D)$; and it isolates the remaining Dirichlet-space question as a scalar-orbit problem.

Several questions remain open.

Open problem 1: Let $\phi \in \text{Aut}(D)$ be parabolic or hyperbolic. Decide whether the operator C_ϕ can be numerically hypercyclic on the Dirichlet space D .

Open problem 2: Let ϕ be a parabolic non-automorphic linear-fractional self-map of D . Determine whether C_ϕ may be numerically hypercyclic on $H^2(D)$ or on $A^2(D)$. Equivalently, clarify whether the known failure of ordinary hypercyclicity in this case also excludes density of numerical orbits.

Open problem 3: Investigate numerical hypercyclicity for weighted composition operators

$$W_{\psi,\phi}f = \psi(f \circ \phi) \quad (6.3)$$

on classical spaces of analytic functions, starting with the Hardy and Bergman spaces.

Overall, the results show that numerical hypercyclicity for composition operators is not governed by a single universal principle. It is ruled out by power boundedness in several classical settings, coincides with ordinary hypercyclicity for automorphic Hardy and Bergman operators, and remains a subtle scalar-orbit question on the Dirichlet space.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Funding

The authors received no specific funding for this work.

Acknowledgments

The authors are grateful to the anonymous referee for the careful reading of the manuscript and for the valuable comments and suggestions, which helped improve the mathematical rigor, clarity, and presentation of the paper.

Data Availability

No datasets were generated or analyzed during the current study.

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