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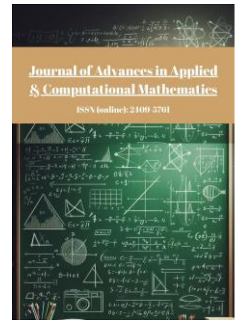

Nonstandard Discretization Scheme in Volterra Integro-differential Equations that Preserves Uniform Asymptotic Stability

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ABSTRACT

We apply a nonstandard discretization scheme to continuous Volterra integro-differential equations and we show that under this discretization, the necessary and sufficient conditions for uniform asymptotic stability of continuous Volterra integro-differential equations are preserved. Our analysis is based on the notion of resolvent. An example is provided as an application to our theory.

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1. Introduction

In this paper we apply a nonstandard discretization scheme to a Volterra integro-differential equation to form a Volterra discrete system. By using resolvent equations, we will show that under the same conditions on some of the coefficients necessary and sufficient conditions for the uniform asymptotic stability of the zero solution are preserved in both systems.

This paper is motivated by the work of [1-3]. In [1], the authors gave a survey of nonstandard schemes for biological models and proposed an open problem for obtaining a nonstandardized scheme for Volterra integro-differential equations that preserve certain qualitative properties of the solutions.

W. Kahan [4] considered the system of differential equations

$$\begin{cases} x'(t) = \alpha x + \beta xy \\ y'(t) = \gamma y + \delta xy \end{cases} \tag{1}$$

and used the discretization scheme

$$\frac{x(t+h)-x(t)}{h} = \frac{\alpha}{2}[x(t+h)+x(t)] + \frac{\beta}{2}[x(t+h)y(t)+x(t)y(t+h)]$$

$$\frac{y(t+h)-y(t)}{h} = \frac{\gamma}{2}[y(t+h)+y(t)] + \frac{\delta}{2}[x(t+h)y(t)+x(t)y(t+h)]$$

to approximate model (1), where the step size h satisfies $0 < h < 1$. A careful examination of [4] by Sanz-Serna [5] revealed that the discretization of equation (1) by Kahan provided a symplectic numerical integration scheme. Next, Mickens [6], using his nonstandard finite difference (NSFD) methods, showed that another unconventional scheme existed for equation (1), such that for the case of Lotka-Volterra equations the neutral stability periodic solutions could be reproduced by this discretization. This work was subsequently extended by Mounim and De Dormale [7] who constructed additional NSFD schemes which greatly improved the accuracy of numerical solutions.

For motivational purposes, consider the differential equation

$$x'(t) = ax(t), \ for\ some\ constant\ a < 0. \tag{2}$$

Then, all solutions $x(t)$ of (2) satisfy

$$x(t) \to 0 as\ t \to \infty.$$

Consider the approximations

$$x'(t) \approx \frac{x(t+h)-x(t)}{\Phi(h)}, \ x(t) \approx \frac{x(t+h)+x(t)}{2}, \tag{3}$$

where $\Phi(h)$ is continuous, $h > 0$ and $\Phi(h) = h + O(h^2)$. Note that the approximations when $\Phi(h) = h + O(h^2)$ are more accurate than the approximations when $\Phi(h) = h$. Applying the approximations given by (3) to equation (2) we obtain the analogous discrete system

$$x(n+1) = \frac{2+a\Phi(h)}{2-a\Phi(h)}x(n), \tag{4}$$

where $x(n+1) = x(t+h)$ and $x(n) = x(t)$. All solutions $x(n)$ of (4) satisfy $x(n) \to 0 as\ n \to \infty$, provided that

$$\left|\frac{2+a\Phi(h)}{2-a\Phi(h)}\right| < 1. \tag{5}$$

Note that condition (5) holds for all $a < 0$ and $\Phi(h) > 0$. Thus, we see that the discretization scheme defined by (3) preserved the stability of the zero solution of (2).

For more on the use of nonstandard discretization, we refer the reader to [1] and [8].

In [2], the authors considered the nonlinear Volterra integro-differential equation

$$x'(t) = ax(t) + \int_0^t B(t,s)f(s,x(s))ds,\ \ t \geq 0. \tag{6}$$

They used

$$x'(t) \approx \frac{x(t+h)-x(t)}{h},\ \ x(t) \approx \frac{x(t+h)+x(t)}{2},\ \ 0 < h < 1 \tag{7}$$

along with

$$\int_0^t B(t,s)f(s,x(s))\ ds \approx \sum_{s=0}^t B(t,s)f(s,x(s)), \tag{8}$$

and arrived at the corresponding discrete Volterra equation,

$$x(n+1) = \frac{2+ah}{2-ah}x(n) + \frac{2h}{2-ah}\sum_{s=0}^n B(n,s)f(s,x(s)),\ \ n \geq 0, \tag{9}$$

where $x(n+1) = x(t+h),\ x(n) = x(t)$ and $0 < h < 1$. We recall that $\Phi(h)$ is required to be continuous with $h > 0$ and $\Phi(h) = h + O(h^2)$. For example, $\Phi(h)$ can be taken as $\Phi(h) = (1-e^{-ah})/h$. By the aid of suitable Lyapunov functionals they showed that under discretizations (7) and (8), stability and boundedness of solutions of (9) were preserved.

In a later note by Mickens [3], the author substituted discretization (8) with

$$\int_0^t B(t,s)f(s,x(s))\ ds \approx \sum_{s=0}^t B(t,s)f(s,x(s))h. \tag{10}$$

The main contribution of this paper is to construct a nonstandard discretization scheme for scalar Volterra integro-differential equations that preserves uniform asymptotic stability. By developing discrete resolvent equations, we prove that stability properties of the continuous system are retained in the discrete counterpart.

We are ready to make the following definition.

Definition 1 A resulting difference equation is said to be consistent with respect to property P under a given discretization scheme with its continuous counterpart if they both exhibit property P under equivalent conditions.

Based on Definition 1, we see that (4) is consistent with respect to asymptotic stability with (2) under discretization (3).

2. Uniform Stability Analysis via the Resolvent Approach

Consider the system of linear Volterra integro-differential equations

$$x'(t) = A(t)x(t) + \int_0^t B(t,s)x(s)ds \tag{11}$$

where A, B are $n \times n$ matrix functions, $A(t)$ is continuous on $[0,+\infty)$, $B(t,s)$ is continuous for $0 \leq s \leq t < +\infty$.

The resolvent equations of (11) are

$$\frac{\partial R(t,s)}{\partial s} = -R(t,s)A(s) - \int_s^t R(t,u)B(u,s)du, \quad R(t,t) = I \tag{12}$$

for $t \geq s \geq 0$, or

$$\frac{\partial R(t,s)}{\partial t} = A(t)R(t,s) + \int_s^t B(t,u)R(u,s)du, \quad R(s,s) = I \tag{13}$$

for $t \geq s \geq 0$, where I is the $n \times n$ identity matrix. Similarly, it was proven in [9] that the resolvent matrix $R(n,s)$ of the Volterra difference equation

$$x(n+1) = A(n)x(n) + \sum_{s=0}^{n} B(n,s)x(s) \tag{14}$$

satisfies the resolvent matrix equation

$$R(n+1,s) = A(n)R(n,s) + \sum_{u=s}^{n} B(n,u)R(u,s) \tag{15}$$

if $s \leq n$, $R(s,s) = I$ and $R(n,s) = 0$ if $n < s$. In [10] it was also proven that the resolvent $R(n,s)$ satisfies

$$R(n,s+1)(A(s) - I) + \sum_{u=s}^{n-1} R(n,u+1)\, B(u,s) + \Delta_s R(n,s) = 0 \tag{16}$$

if $s \leq n$, $R(n,n) = I$ and $R(n,s) = 0$ if $n < s$, where $\Delta_s R(n,s) = R(n,s+1) - R(n,s)$.

In this paper we limit our discussion to the scalar Volterra integro-differential equation

$$x'(t) = a(t)x(t) + \int_0^t B(t,s)\, x(s)ds, \tag{17}$$

where a is a given function and $B(t,s)$ is continuous for $0 \leq s \leq t < +\infty$. Then it is clear that (12) and (13) hold for (17) by replacing $A(t)$ with a and considering $B(t,s)$ and $R(t,s)$ as scalar functions with $R(s,s) = 1$.

By considering the discretization scheme (3) for

$$x'(t) = a(t)x(t)$$

and by approximating the integral term with

$$\int_0^t B(t,s)\, x(s)\; ds \approx \sum_{s=0}^{t} B(t,s)x(s)h, \tag{18}$$

we arrive at the corresponding discrete Volterra equation,

$$\frac{x(n+1)-x(n)}{\Phi(h)} = a(n)\frac{x(n+1)+x(n)}{2} + h\sum_{s=0}^{n} B(n,s)x(s), \quad n \geq 0,$$

or

$$x(n+1) - x(n) = \frac{a(n)\Phi(h)}{2}x(n+1) + \frac{a(n)\Phi(h)}{2}x(n) + h\Phi(h)\sum_{s=0}^{n} B(n,s)\, x(s), \quad n \geq 0,$$

or

$$\frac{2-a(n)\Phi(h)}{2}x(n+1) = \frac{2+a(n)\Phi(h)}{2}x(n)_+ h\Phi(h)\sum_{s=0}^{n} B(n,s)\, x(s), \quad n \geq 0,$$

or

$$x(n+1) = \frac{2+a(n)\Phi(h)}{2-a(n)\Phi(h)}x(n) + \frac{2h\Phi(h)}{2-a(n)\Phi(h)}\sum_{s=0}^{n} B(n,s)x(s),\; n \geq 0, \tag{19}$$

where $x(n+1) = x(t+h)$, $x(n) = x(t)$ and $h > 0$. The study of Volterra difference systems is important since they play a major role in the fields of numerical analysis, control theory and computer science. Thus, finding a discretization scheme under which Eq. (19) is consistent with Eq. (17) is important.

Let $t_0 \geq 0$, then for each given bounded initial function $\varphi: [0,t_0] \to R$, we say $x(t) = x(t,t_0,\varphi)$ is a solution of (17) if it satisfies (17) for $t \geq t_0$ and $x(t) = \varphi(t)$ for $t \in [0,t_0]$. For $\varphi: [0,t_0] \to R$, we define $||\varphi|| = sup\{\,|\varphi(s)|: 0 \leq s \leq t_0\}$. In a similar fashion a solution $x(t) = x(t,t_0,\varphi)$ of (19) can be easily defined.

Before we proceed any further, we state what it means for the zero solution of (17) and (19) to be uniformly stable and uniformly asymptotically stable:

Definition 2 The zero solution of (17) is stable if for each $\varepsilon > 0$ and each $t_0 \geq 0$, there exists a $\delta = \delta(\varepsilon, t_0) > 0$ such that if φ is a given bounded continuous initial function with $\|\varphi\| < \delta$, then $|x(t, t_0, \varphi)| < \varepsilon$ for all $t \geq t_0$. The zero solution of (17) is uniformly stable (US) if δ is independent of t_0.

Definition 3 The zero solution of (17) is uniformly asymptotically stable (UAS) if it is US and there exists a $\delta_0 > 0$ with the property that for each $\varepsilon > 0$, there exists $T = T(\varepsilon) > 0$ such that $[t_0 \geq 0,\ \|\varphi\| < \delta_0, t \geq T + t_0]]$ imply $|x(t, t_0, \varphi)| < \varepsilon$.

In a similar manner, one can easily state Definitions 2 and 3 for (19). For more on the stability of Volterra integro-differential equations or Volterra difference equations, we refer the reader to [11] and [12].

Now, we will approximate the corresponding resolvent equation for the equation (11). We will use the discretization scheme

$$\frac{\partial R(t,s)}{\partial s} \approx \frac{R(t,s+h)-R(t,s)}{\Phi(h)},\ \ R(t,s) \approx \frac{R(t,s+h)+R(t,s)}{2}, \tag{20}$$

where $\Phi(h)$ is continuous, $h > 0$ and $\Phi(h) = h + O(h^2)$, and

$$\int_s^t R(t,u)\, B(u,s)\ ds \approx \sum_{u=s}^{t-1} R(t, u+1)\, B(u,s)h \tag{21}$$

to discretize (12) when it is in the scalar form to arrive at

$$\frac{R(n,s+h)-R(n,s)}{\Phi(h)} = -\frac{R(n,s+h)+R(n,s)}{2} A(n) - h\sum_{u=s}^{n-1} R(n,u+1)\, B(u,s),$$

$n \geq s$, or

$$\begin{aligned} R(n,s+h) - R(n,s) = \quad & -R(n,s+h)\frac{\Phi(h)}{2}A(n) - R(n,s)\frac{\Phi(h)}{2}A(n) \\ & -h\Phi(h)\sum_{u=s}^{n-1} R(n,u+1)\, B(u,s), \quad n \geq s, \end{aligned}$$

or

$$\frac{2+\Phi(h)A(n)}{2} R(n,s+h) - \frac{2-\Phi(h)A(n)}{2} R(n,s) + h\Phi(h)\sum_{u=s}^{n-1} R(n,u+1)\, B(u,s) = 0,$$

$n \geq s$, or

$$\frac{2+A(n)\Phi(h)}{2-A(n)\Phi(h)} R(n,s+1) + \frac{2h\Phi(h)}{2-a(n)\Phi(h)}\sum_{u=s}^{n-1} R(n,u+1)\, B(u,s) - R(n,s) = 0 \tag{22}$$

if $s \leq n$, $R(n,n) = 1$ and $R(n,s) = 0$ if $n < s$.

We note that in obtaining the final form of (22), we used the notation $R(t,s) = R(n,s)$ and $R(t,s+h) = R(n,s+1)$.

Throughout this paper we assume that $2 - a\Phi(h) \neq 0$, which is satisfied whenever $a < 0$ and $\Phi(h) > 0$. The next lemma is needed for the main results.

Lemma 1 Let $\varphi(t)$ be a given bounded initial function. Then $x(n)$ is a solution of (19) if and only if

$$x(n) = R(n,n_0)\varphi(n_0) + \sum_{s=n_0}^{n-1}\sum_{u=0}^{n_0-1} \frac{2h\Phi(h)}{2-a(s)\Phi(h)} R(n,s+1)B(s,u)\varphi(u),\ \ n \geq 0, \tag{23}$$

where $R(n,s)$ satisfies (22).

Proof. To simplify notation we let $C=\frac{2+a(s)\Phi(h)}{2-a(s)\Phi(h)}-1$. Also we let $x(n)$ be a solution of (19) and $R(n,s)$ satisfies (22). By summing the expression

$$\Delta(R(n,s)x(s))=R(n,s+1)\Delta x(s)+(\Delta_s R(n,s))x(s)$$

from n_0 to $n-1$ we get

$$\begin{aligned}
x(n)-R(n,n_0)\varphi(n_0) &= \sum_{s=n_0}^{n-1}[R(n,s+1)\,\Delta x(s)+(\Delta_s R(n,s))x(s)]\\
&= \sum_{s=n_0}^{n-1} R(n,s+1)\,[Cx(s)\\
&+\frac{2h\Phi(h)}{2-a(s)\Phi(h)}\sum_{u=0}^{s} B(s,u)\,x(u)]\\
&+\sum_{s=n_0}^{n-1}(\Delta_s R(n,s))\,x(s)\\
&= \sum_{s=n_0}^{n-1} R(n,s+1)\,[Cx(s)\\
&+\frac{2h\Phi(h)}{2-a(s)\Phi(h)}(\sum_{u=0}^{n_0-1} B(s,u)\,\varphi(u)\\
&+\sum_{u=n_0}^{s} B(s,u)\,x(u))]+\sum_{s=n_0}^{n-1}\big(\Delta_s R(n,s)\big)x\,(s)
\end{aligned} \tag{24}$$

By noting that

$$\sum_{s=n_0}^{n-1}\sum_{u=n_0}^{s} R(n,s+1)\,B(s,u)x(u)=\sum_{s=n_0}^{n-1}\sum_{u=s}^{n-1} R(n,u+1)\,B(u,s)x(s),$$

(24) reduces to,

$$\begin{aligned}
x(n)-R(n,n_0)\varphi(n_0)-\sum_{s=n_0}^{n-1}\frac{2h\Phi(h)}{2-a(s)\Phi(h)}R(n,s+1)\sum_{u=0}^{n_0-1} B(s,u)\,\varphi(u)\\
=\sum_{s=n_0}^{n-1}[R(n,s+1)\frac{2+a(s)\Phi(h)}{2-a(s)\Phi(h)}-R(n,s)\\
+\frac{2h\Phi(h)}{2-a(s)\Phi(h)}\sum_{u=s}^{n-1} R(n,u+1)\,B(u,s)]x(u)
\end{aligned} \tag{25}$$

Since $R(n,s)$ satisfies (22), we have the right side of (25) equal to zero and hence (23) .

Through this paper we assume that

$$|B(\cdot,\cdot)| \ is \ bounded. \tag{26}$$

Theorem 1 The zero solution of (19) is consistent with respect to uniform stability under the discretization scheme (3), (18), (20), and (21) with its continuous counterpart (17) if and only if

$$\sup_{t\geq t_0\geq 0}\{|R(t,t_0)|+\int_0^{t_0}|\int_{t_0}^{t}R(t,s)B(s,u)ds\,|\,du\}<+\infty, \tag{27}$$

Proof. The proof for the continuous equation given by (17) follows from [13, Theorem 1]. Now, by (26) and (27) there exist a constant $E > 0$ such that

$$\sup_{n\geq n_0\geq 0}\left\{|R(n,n_0)|+\sum_{u=0}^{n_0-1}\sum_{s=n_0}^{n-1}\left|\frac{2h\Phi(h)}{2-a(s)\Phi(h)}\right|R(n,s+1)B(s,n)|\right\}<E. \tag{28}$$

Thus, a change of the order of summation in (23) yields,

$$\begin{aligned}|x(n)| &\leq \left\{|R(n,n_0)|+\sum_{u=0}^{n_0-1}\ \ |\sum_{s=n_0}^{n-1}\ \ \frac{2h\Phi(h)}{2-a(s)\Phi(h)}R(n,s+1)B(s,u)|\right\}||\varphi|| \\ &\leq E||\varphi||. \end{aligned}\tag{29}$$

Thus (29) implies the zero solution of (19) is uniformly stable.

Suppose that the zero solution of (19) is US. Then for $\varepsilon = 1$, there exists a $\delta > 0$ such that $[n_0 \geq 0, \varphi \in C(n_0), ||\varphi|| \leq \delta, n \geq n_0]$ implies $|x(n,n_0,\varphi)| < 1$. Let m be a positive integer and define the sequence of functions φ_m by

$$\varphi_m(u) = va^{-m(n_0-u)} \ on \ 0 \leq u \leq n_0. \tag{30}$$

Let $\psi_m(u) = \delta v a^{-m(n_0-u)}/2$ for $0 \leq u \leq n_0$. Then, $|\psi_m(u)| \leq \delta/2$. Hence we have $|x(n,n_0,\psi_m(s))| < \varepsilon$. It is clear from (30) that $\varphi_m(n_0) = v$ and $|\varphi_m(s)| \leq 1$ for $0 \leq s \leq n_0$. Thus, from (23) we have:

$$\begin{aligned}|R(n,n_0)|\frac{\delta}{2} &\leq |x(n,n_0,\psi_m)| \\ &+\frac{\delta}{2}\sum_{s=n_0}^{n-1}\left|\frac{2h\Phi(h)}{2-a(s)\Phi(h)}R(n,s+1)\sum_{u=0}^{n_0-1}B(s,u)a^{-m(n_0-u)}\right| \\ &\leq 1+\frac{\delta}{2}\sum_{s=n_0}^{n-1}\left|\frac{2h\Phi(h)}{2-a(s)\Phi(h)}R(n,s+1)\sum_{u=0}^{n_0-1}B(s,u)a^{-m(n_0-u)}\right|.\end{aligned}\tag{100.1}$$

Now, for fixed n, we have:

$$\left|\sum_{s=n_0}^{n-1}(n,s+1)\,R\sum_{u=0}^{n_0-1}B(s,u)a^{-m(n_0-u)}\right| \to 0 \ as \ m \to \infty.$$

Now (100.1) yields

$$|R(n,n_0)| \leq \tfrac{2}{\delta}. \tag{31}$$

Next, let $\varphi \in C(n_0)$ with $||\varphi|| < 1$. Define $\psi = \delta\varphi$. Then $||\psi|| < \delta$. Thus, by the definition of δ, we have $|x(n, n_0, \psi)| < 1$ for all $n \geq n_0$. It follows from (19) and (23) that

$$\left|\sum_{s=n_0}^{n-1} \frac{2h\Phi(h)}{2 - a(s)\Phi(h)} R(n, s+1) \sum_{u=0}^{n_0-1} B(s,u)\psi(u)\right|$$

$$\leq |x(n, n_0, \psi)| + |R(n, n_0)||\psi(n_0)|$$

$$\leq |x(n, n_0, \psi)| + |R(n, n_0)|\ ||\psi(n_0)|| \leq 3.$$

Hence,

$$\left|\sum_{u=0}^{n_0-1} \sum_{s=n_0}^{n-1} R(n, s+1)\, \phi(u)\right| \leq \frac{1}{\delta}\left|\sum_{n=0}^{n_0-1} \sum_{s=n_0}^{n-1} R(n, s+1)\, \psi(n)\right| \leq \frac{3}{\delta}\left|\frac{2 - a\Phi(h)}{2h\Phi(h)}\right|$$

for $n \geq n_0$ and the proof is complete.

The next theorem is about uniform asymptotic stability of the zero solution.

Theorem 2 The zero solution of (19) is consistent with respect to uniform asymptotic stability under the discretization scheme (3), (18), (20), and (21) with its continuous counterpart (17) if and only if (27) holds and

$$\{|R(t, t_0)| + \textstyle\int_0^{t_0} |\int_{t_0}^{t} R(t, s)\, B(s, u) ds|\, du\} \to 0 \tag{32}$$

as $n - n_0 \to +\infty$ uniformly,

Proof. The proof for the continuous equation given by (17) follows from [13, Theorem 2.3] As for (17), by Theorem 1, the zero solution is obviously US. Let $B_1 > 0$ be given and $\varphi \in C(n_0)$ on $0 \leq s \leq n_0$ with $||\varphi|| \leq B_1$. Then, it follows from using (26) and (27) in (23) that,

$$|x(n)| \leq \left\{|R(n, n_0)| + \sum_{u=0}^{n_0-1} \sum_{s=n_0}^{n-1} \left|\frac{2h\Phi(h)}{2 - a(s)\Phi(h)} R(n, s+1) B(s, u)\right|\right\} ||\varphi||$$

$$\leq \left[|R(n, n_0)| + \sum_{u=0}^{n_0-1} \sum_{s=n_0}^{n-1} \left|\frac{2h\Phi(h)}{2 - a(s)\Phi(h)} R(n, s+1) B(s, u)\right|\right] B_1.$$

From (32), it follows that for any $\varepsilon > 0$, there exists a constant $T > 0$ such that

$$\left[|R(n, n_0)| + \sum_{u=0}^{n_0-1} \sum_{s=n_0}^{n-1} \left|\frac{2h\Phi(h)}{2 - a(s)\Phi(h)} R(n, s+1) B(s, u)\right|\right] < \frac{\varepsilon}{B_1}$$

for all $n \geq T + n_0$. Thus, $|x(n)| < \varepsilon$ for all $n \geq T + n_0$. This implies that the zero solution of (1) is UAS.

Conversely, suppose that the zero solution of (17) is UAS. Then it is US. Let $\varphi \in C(n_0)$ with $||\varphi|| \leq 1$. Then, for any $\varepsilon > 0$, there exists $T > 0$ such that $|x(n, n_0, \varphi)| < \varepsilon$ for $n \geq T + n_0$. By making use of (100.1) and by the argument of Theorem 1, we have $|R(n, n_0)| < \varepsilon$ for all $n \geq T + n_0$. Now using (19), we get

$$\sum_{u=0}^{n_0-1} \left(\sum_{s=n_0}^{n-1} \left|\frac{2h\Phi(h)}{2 - a(s)\Phi(h)} R(n, s+1) B(s, u)\right.\right) \varphi(u)|$$

$$\leq |x(n, n_0, \varphi)| + |R(n, n_0)| < 2\varepsilon$$

for all $n \geq T + n_0$. This implies

$$\sum_{u=0}^{n_0-1} \sum_{s=n_0}^{n-1} |\frac{2h\Phi(h)}{2 - a(s)\Phi(h)} R(n, s+1)B(s,u)| < 2\varepsilon$$

for all $n \geq T + n_0$. Therefore,

$$\left\{ |R(n, n_0)| + \sum_{u=0}^{n_0-1} \sum_{s=n_0}^{n-1} |\frac{2h\Phi(h)}{2 - a(s)\Phi(h)} R(n, s+1)B(s,u)| \right\} \to 0$$

as $n - n_0 \to +\infty$ uniformly, and this completes the proof.

3. Applications and Lyapunov Analysis

In this section, we use Lyapunov functional in terms of the resolvent to verify the condition given in Theorem 2.

Theorem 3 Consider the scalar equation

$$x^{'}(t) = a(t)x(t) + \int_0^t b(t,s)x(s)\, ds. \tag{33}$$

Suppose that there are positive constants α, h, and K with $K > 1$ satisfying the following conditions for $t \geq 0$:

1. $a(t) + K \rightleftarrows \int_0^t |b(t,s)|\, ds \leq 0$,
2. For each $\alpha > 0$, there exists $h > 0$ such that $\int_t^{t+h} |a(s)|\, ds \geq \alpha$,
3. $\frac{1}{|a(t)|} \int_0^{t_0} |b(t,s)|\, ds \to 0$ as $t - t_0 \to +\infty$ uniformly on $\{t | a(t) \neq 0\}$.

The zero solution of (19) is consistent with respect to uniform asymptotic stability under the discretization scheme (3), (18), (20), and (21) with its continuous counterpart (33).

Proof. By the assumptions (i)-(iii), we get the following.

- $-a(n) + K(\rightleftarrows 1 - \sum_{s=0}^{n} |b(n,s)|) > 0$,
- For each $\gamma > 0$, there exists $h > 0$ such that $\sum_{s=n}^{n+h-1} |a(s)| \geq \gamma$,
- $\frac{1}{|\frac{2+a(n)\Phi(h)}{2h\Phi(h)}|} \sum_{s=0}^{n_0} |b(n,s)| \to 0$ as $n - n_0 \to +\infty$

uniformly on $\{n | 2 + a(n)\Phi(h) \neq 0\}$, and

- $\sum_{u=0}^{\infty} |b(u,s)| \leq B$.

Define the discrete Lyapunov functional, $V(s)$ on $[0, n-1]$ by

$$V(s) = |R(n,s)| + \sum_{u=s}^{n-1} \sum_{v=0}^{s-1} |\frac{2h\Phi(h)}{2 - a(u)\Phi(h)}| |R(n, u+1)|\ |b(u,v)|,$$

where $R(n,s)$ is the resolvent of (19) given by

$$\frac{2+a(n)\Phi(h)}{2-a(n)\Phi(h)}R(n,s+1)+\frac{2h\Phi(h)}{2-a(n)\Phi(h)}\sum_{u=s}^{n-1}R(n,u+1)\,b(u,s)-R(n,s)=0,$$

if $s\le n$, $R(n,n)=1$ and $R(n,s)=0$ if $n<s$.

Then, using (i) we have:

$$\begin{aligned}\Delta V(s) &= |R(n,s+1)|-|R(n,s)| \\ &+\sum_{u=s+1}^{n-1}\sum_{v=0}^{s}\;|\frac{2h\Phi(h)}{2-a(u)\Phi(h)}||R(n,u+1)||b(u,v)| \\ &-\sum_{u=s}^{n-1}\sum_{v=0}^{s-1}\;|\frac{2h\Phi(h)}{2-a(u)\Phi(h)}||R(n,u+1)||b(u,v)| \\ &\ge\left(-|\frac{2+a(s)\Phi(h)}{2-a(s)\Phi(h)}|+1\right)|R(n,s+1)| \\ &-|\frac{2h\Phi(h)}{2-a(s)\Phi(h)}||R(n,s+1)||b(s,s)| \\ &-|R(n,s+1)|\sum_{v=0}^{s-1}\;|\frac{2h\Phi(h)}{2-a(v)\Phi(h)}||b(s,v)| \\ &=\left(1-|\frac{2+a(s)\Phi(h)}{2-a(s)\Phi(h)}|-\sum_{v=0}^{s}\;|\frac{2h\Phi(h)}{2-a(v)\Phi(h)}||b(s,v)|\right)|R(n,s+1)|\end{aligned}\tag{100.2}$$

Since $\Phi(h)\to 0$ as $h\to 0$ and $a<0$, we have

$$|\frac{2+a(s)\Phi(h)}{2-a(s)\Phi(h)}|<1 \quad and \quad |\frac{2+a(s)\Phi(h)}{2-a(s)\Phi(h)}|\to 1 \quad as \quad h\to 0,$$

$$\sum_{v=0}^{s}\;|\frac{2h\Phi(h)}{2-a(v)\Phi(h)}||b(s,v)|\to 0 \quad as \quad h\to 0,$$

we can choose $h>0$ small enough so that

$$1-|\frac{2+a(s)\Phi(h)}{2-a(s)\Phi(h)}|-\sum_{v=0}^{s}\;|\frac{2h\Phi(h)}{2-a(v)\Phi(h)}||b(s,v)|>0.$$

For such choice of h, there is a positive constant $K_1<1$ such that

$$1-|\frac{2+a(s)\Phi(h)}{2-a(s)\Phi(h)}|-\sum_{v=0}^{s}\;|\frac{2h\Phi(h)}{2-a(v)\Phi(h)}||b(s,v)|\ge\left(\frac{1}{K_1}-1\right)|\frac{2+a(s)\Phi(h)}{2-a(s)\Phi(h)}|.$$

Thus the inequality (100.2), becomes

$$\Delta V(s)\ge\left(\frac{1}{K_1}-1\right)|\frac{2+a(s)\Phi(h)}{2-a(s)\Phi(h)}||R(n,s+1)|. \tag{34}$$

Then we have $\Delta V(s)>0$.

This yields that for $n\ge n_0\ge 0$, $V(n_0)\le V(n)=|R(n,n)|=1$. That is,

$$|R(n,n_0)| + \sum_{u=n_0}^{n-1} \sum_{v=0}^{n_0-1} |\frac{2h\Phi(h)}{2-a(v)\Phi(h)}||R(n,u+1)||b(u,v)| \leq 1. \tag{35}$$

Hence (28) is satisfied. Thus condition (28) is satisfied and by Theorem 1 the zero solution of (19) is US.

By summing (34) from 0 to $n-1$, we obtain

$$(\frac{1}{K_1}-1)\sum_{s=0}^{n-1} |\frac{2+a(s)\Phi(h)}{2-a(s)\Phi(h)}||R(n,s+1)| \leq V(n)-V(0) \leq 1 \tag{36}$$

or

$$\sum_{s=0}^{n-1} |\frac{2+a(s)\Phi(h)}{2-a(s)\Phi(h)}||R(n,s+1)| \leq \frac{K_1}{1-K_1} =: D.$$

By (iii), for any $\varepsilon > 0$ there exists $N_1 > 0$ such that for $u \geq N_1 + s - 1$ implies

$$\sum_{v=0}^{s-1} |b(u,v)| \leq \frac{\varepsilon}{(3+B)D}|\frac{2+a(u)\Phi(h)}{2h\Phi(h)}|.$$

Thus, for $n \geq N_1 + s - 1$ we have

$$\begin{aligned}
&\sum_{u=s}^{s+N_1-1} |\frac{2h\Phi(h)}{2-a(u)\Phi(h)}||R(n,u+1)|\sum_{v=0}^{s-1} |b(u,v)| \\
&= \sum_{u=s}^{n-1} |\frac{2h\Phi(h)}{2-a(u)\Phi(h)}||R(n,u+1)|\sum_{v=0}^{s-1} |b(u,v)| \\
&+ \sum_{u=s+N_1}^{n-1} |\frac{2h\Phi(h)}{2-a(u)\Phi(h)}||R(n,u+1)|\sum_{v=0}^{s-1} |b(u,v)| \\
&\leq \frac{\varepsilon}{(3+B)D}\sum_{u=s+N_1}^{n-1} |\frac{2h\Phi(h)}{2-a(u)\Phi(h)}||R(n,u+1)||\frac{2+a(u)\Phi(h)}{2h\Phi(h)}| \\
&+ \sum_{u=s}^{s+N_1-1} |\frac{2h\Phi(h)}{2-a(u)\Phi(h)}||R(n,u+1)|\sum_{v=0}^{s-1} |b(u,v)| \\
&\leq \frac{\varepsilon}{3+B} + \sum_{u=s}^{s+N_1-1} |\frac{2h\Phi(h)}{2-a(u)\Phi(h)}||R(n,u+1)|\sum_{v=0}^{s-1} |b(u,v)|.
\end{aligned} \tag{37}$$

Let $\beta = \frac{K}{1-K}$ and $\alpha = \frac{3+B}{\varepsilon\beta}$. By (ii), there exists an $h > 0$ such that $\sum_{v=s}^{s+h-1} |a(v)| \geq \alpha$, and

$$|R(n,n_s+1)|\beta \sum_{v=s}^{s+h-1} |a(v)| \leq \beta \sum_{u=s}^{s+h-1} |R(n,u+1)|\,|a(u)|,$$

for $n_s \in [s, s+h-1]$ and $n \geq s+h$, where

$$|R(n,n_s+1)| = \min_{s \leq u \leq s+h-1} |R(n,u+1)|.$$

Using (36) in the above inequality we arrive at

$$|R(n,n_s+1)| \leq \frac{1}{\beta\sum_{v=s}^{s+h-1} \quad |a(v)|} < \frac{\varepsilon}{3+B}.$$

Choose $N > 1$ so that $\frac{\beta N\varepsilon}{3+B} > 1$. For each $n_0 \geq 0$ and $n \geq n_0 + (N+1)(N_1+h-1)$, define $\{n_j\}$ with

$$n(j-1) + N_1 \leq n_j \leq n(j-1) + N_1 + h - 1, \quad j = 1,2,3.\ldots.N$$

such that

$$|R(n,n_j+1)| < \frac{\varepsilon}{3+B}. \tag{38}$$

It follows that $n_N \leq n_0 + N(N_1+h-1)$ and by (36) we arrive at

$$\sum_{j=1}^{N} \Big(\sum_{u=n_j}^{n_j+N_1-1} \beta|R(n,u+1)||a(u)|\Big) \leq \sum_{u=n_0}^{n-1} \beta|R(n,u+1)||a(u)| \leq 1.$$

Since $\frac{\beta N\varepsilon}{3+B} > 1$, it follows from the above inequality there exists $n_k, 1 \leq k \leq N$ such that

$$N\sum_{u=n_k}^{n_k+N_1-1} \beta|R(n,u+1)||a(u)| \leq 1.$$

Or,

$$\sum_{u=n_k}^{n_k+N_1-1} |R(n,u+1)||a(u)| < \frac{\varepsilon}{3+B}. \tag{39}$$

Since $V(s)$ is increasing, we have

$$V(n_k) \leq V(n_k+1).$$

Hence, using (37)-(38) and (vi) we arrive at

$$|R(n,n_k)| + \sum_{u=n_k}^{n-1}\sum_{v=0}^{n_k-1} |R(n,u+1)|\ |b(u,v)|$$

$$\leq |R(n,n_k+1)| + \sum_{u=n_k}^{n-1}\sum_{v=0}^{n_k-1} |R(n,u+1)|\ |b(u,v)|$$

$$+ \sum_{u=n_k+1}^{n-1} |R(n,u+1)|\ |b(u,n_k)|$$

$$\leq \frac{\varepsilon}{3+B} + \frac{2\varepsilon}{3+B} + \frac{B\varepsilon}{3+B} = \varepsilon.$$

This yields

$$|R(n,n_0)| + \sum_{u=n_0}^{n-1}\sum_{v=0}^{n_0-1} |R(n,u+1)|\ |b(u,v)| = V(n_0) \leq V(n_k) < \varepsilon \tag{40}$$

for $n \geq n_0 + (N+1)(N_1 + h - 1) \geq n_k, N > \frac{3+B}{\beta\varepsilon}$. Hence condition (2) is satisfied. Next, for $n \geq n_0 + (N+1)(N_1 + h - 1) \geq n_k$ we have by using condition (iv) in (40).

$$
\begin{aligned}
&|R(n,n_0)| + \sum_{u=n_0}^{n-1} |R(n,u+1)|\lambda(u) \\
&\leq |R(n,n_0)| + \sum_{u=n_0}^{n-1} |R(n,u+1)| \sum_{v=0}^{n_0-1} |b(u,v)| \\
&\leq |R(n,n_0)| + \sum_{u=n_0}^{n-1} \sum_{v=0}^{n_0-1} |R(n,u+1)||b(u,v)| \\
&< \varepsilon. \qquad (41)
\end{aligned}
$$

Hence the zero solution of (33) is UAS by Theorem 2.7, where we are assuming (26) with $B(t,s)$ is being replaced with $b(t,s)$.

Example 1 Let $t_0 > 0$ be fixed. Consider the equation

$$
x'(t) = -3(1+(t-t_0)^2) + \frac{1}{1+t^2}\int_0^t \frac{s}{(1+s^2)^2} x(s)ds, \quad t \geq 0. \qquad (42)
$$

Here

$$
a(t) = -3(1+(t-t_0)^2),
$$

$$
b(t,s) = \frac{s}{(1+t^2)(1+s^2)^2}, \quad t,s \geq 0.
$$

Take $K = 2$. Then

$$
\begin{aligned}
\int_0^t |b(t,s)|ds &= \frac{1}{1+t^2}\int_0^t \frac{s}{(1+s^2)^2} ds \\
&= \frac{1}{2(1+t^2)}\int_0^t \frac{d(1+s^2)}{(1+s^2)^2} \\
&= -\frac{1}{2(1+t^2)}\frac{1}{1+s^2}\Big|_{s=0}^{s=t} \\
&= -\frac{1}{2(1+t^2)}\left(\frac{1}{1+t^2} - 1\right) \\
&= -\frac{1}{2(1+t^2)^2} + \frac{1}{2(1+t^2)}, \quad t \geq 0.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
a(t) + K\int_0^t |b(t,s)|ds &= -3(1+(t-t_0)^2) + 2\left(-\frac{1}{2(1+t^2)^2} + \frac{1}{2(1+t^2)}\right) \\
&= -3(1+(t-t_0)^2) - \frac{1}{(1+t^2)^2} + \frac{1}{1+t^2} \\
&\leq 0, \quad t \geq 0,
\end{aligned}
$$

i.e., (i) of Theorem 3 holds. Next, fix $\alpha > 0$ and $t \geq 0$. Take $h = \frac{\alpha}{2}$. By the mean value theorem, it follows that there exists a $\xi \in (t, t+h)$ such that

$$\begin{aligned}\int_t^{t+h} |a(s)|ds &= |a(\xi)|h \\ &= 3(1 + (\xi - t_0)^2)h \\ &\geq 3h \\ &= \frac{3\alpha}{2} \\ &> \alpha.\end{aligned}$$

Thus, (ii) of Theorem 3 holds. Moreover, using the above computations, we obtain

$$\begin{aligned}\frac{1}{|a(t)|}\int_0^t |b(t,s)|\,ds &= \frac{1}{3(1 + (t - t_0)^2)}\left(\frac{1}{2(1 + t_0^2)} - \frac{1}{2(1 + t_0^2)^2}\right) \\ &\to 0 \quad as \quad t - t_0 \to \infty,\end{aligned}$$

uniformly on $\{t | a(t) \neq 0\}$. Therefore all conditions of Theorem 3 hold. By considering the approximations (3) for $\Phi(h) = h + h^2$, we find the equation

$$\begin{aligned}x(n+1) \quad = \quad & \frac{2 - 3(1 + (n - t_0)^2)(h + h^2)}{2 + 3(1 + (n - t_0)^2)(h + h^2)} x(n) \\ & + \frac{2h(h + h^2)}{2 + 3(1 + (n - t_0)^2)(h + h^2)} \sum_{s=0}^{n} \frac{s}{(1 + n^2)(1 + s^2)^2} x(s), \quad n \geq 0.\end{aligned} \tag{43}$$

In the figure below are shown the solutions of (43) for $t_0 = 2$.

4. Conclusion

In this paper, we have developed and analyzed a nonstandard discretization scheme for Volterra integro-differential equations that preserves the uniform asymptotic stability of the continuous system. By employing a discretization based on the function $\Phi(h) = h + O(h^2)$ and introducing appropriate approximations for both the derivative and the integral terms, we constructed a discrete analogue that remains consistent with its continuous counterpart in the sense of stability preservation. The analysis was performed through the resolvent approach, which provided a powerful framework to establish necessary and sufficient conditions for uniform stability and uniform asymptotic stability.

Our main theoretical results demonstrate that, under mild continuity and monotonicity assumptions on the kernel $B(t,s)$, the discrete resolvent equation retains the essential qualitative features of the continuous resolvent. Specifically, Theorems 1 and 2 show that both the uniform stability and uniform asymptotic stability of the zero solution are preserved by the proposed discretization scheme. Moreover, by constructing an appropriate discrete Lyapunov functional in Theorem 3, we verified that the uniform asymptotic stability of the zero solution is guaranteed when the continuous system satisfies analogous inequalities involving the coefficients $a(t)$ and $b(t,s)$. This parallelism between the continuous and discrete cases establishes the consistency of the proposed nonstandard scheme with respect to the fundamental stability properties.

The results obtained here strengthen the connection between nonstandard finite difference theory and the stability analysis of Volterra-type systems. The discretization framework outlined in this work can serve as a prototype for other classes of integro-differential equations, including nonlinear, time-varying, or delay systems.

Future work may explore generalizations to multi-dimensional or fractional-order Volterra systems, as well as applications in numerical simulations where long-term qualitative behavior such as boundedness and periodicity must be preserved. Additionally, extending the present results to systems with stochastic or impulsive effects represents a promising direction for further research.

Conflicts of Interest

The authors declare no conflicts of interest.

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