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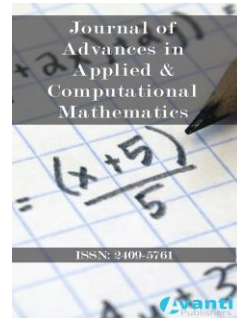

A Conformable Inverse Problem with Constant Delay

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ABSTRACT

This paper aims to express the solution of an inverse Sturm-Liouville problem with constant delay using a conformable derivative operator under mixed boundary conditions. For the problem, we stated and proved the specification of the spectrum. The asymptotics of the eigenvalues of the problem was obtained and the solutions were extended to the Regge-type boundary value problem. As such, a new result, as an extension of the classical Sturm-Liouville problem to the fractional phenomenon, has been achieved. The uniqueness theorem for the solution of the inverse problem is proved in different cases within the interval $(0, \pi)$. The results in the classical case of this problem can be obtained at $\alpha = 1$.

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1. Introduction

As differential equations are used to model real-life problems in sciences, technology, social sciences, etc, the Sturm-Liouville equation, which is a special case of a second-order differential equation, plays a vital role in the literature both in classical and fractional cases. The fractional derivative approach is such a vital tool in which certain phenomenon, that can not be or is very difficult to analyze in the classical case, can easily be analyzed and expressed. There are many fractional derivative approaches such as Riemann-Liouville, Caputo, M-derivative, Grunwald-Letnikov, Weyl, etc, but each has its shortcomings [1-3]. Looking into those shortcomings associated with the most popular fractional differentiation approaches, Khalil, *et al.* [4] established a new fractional derivatives approach which turned out to be easy in evaluations and satisfied most of the properties of differentiation and named it *Conformable Fractional Derivative*. This new approach was criticized, affirmed, and further developed and is in use by many authors [5-9]. As this article involves an inverse problem in the Sturm-Liouville Problem (SLP) in fractional case, there are many studies on the fractional SLP that are being progressed as can be seen in [10-19].

There is an inverse problem in a parameter identification problem of partial differential equations, which involves finding the unknown parameter (usually $p(x)$) from some observed data from the situation under consideration in the system. Under this, many authors designed many inversion strategies that appropriately describe and solve many different inverse problems, details on these can be obtained from [20-25], but the inverse problem in SLP, deals with the concept of finding the potential $q(x)$ and the constants in the conditions in the differential SLP by using the spectral parameters. Ambarzumyan theorem gives the first results of inverse SLP [26], it says that if the spectrum of SLP under Neumann boundary conditions is $\{n^2\}$, $n \geq 0$, then the potential function will be zero, ($q(x) = 0$). The authors in [27, 28] gave some results, in various cases, on this theory. The inverse SLP has been under discussion for a long time by many researchers and so many results have been obtained by many authors as in [29-38].

Inverse SLP under fractional derivative operator is now one of the current research fields, researchers are crescively expanding their studies in the area and many results in different problems were obtained as detailed in [39-42]. There are differential equations with delay in various mathematical problems and applications which produces vital changes in the quality of the studies on the corresponding inverse problems of spectral analysis. The methods of transformation operator, spectral mappings, etc, are the standard methods of solving an inverse SLP without delay (differential operators), but these methods do not work for operators with delay, as such some researchers constructed new approaches for the latter general spectral theory. The authors in [43] consider the Sturm-Liouville differential equation with a large constant delay, that is $a \in \left[\frac{\pi}{2}, \pi\right)$, and generated an effective algorithm for solving the problem, also the authors in [44] studied the nonlinear inverse problem and obtained the Properties of their spectral characteristics. Most of the studies in inverse SLP focus on the situation with zero initial function with the assumption that the potential $q(x)$ vanishes on the corresponding subinterval but the authors in [45] waived that assumption in favor of a continuously matching initial function, which leads to appearing an additional term with frozen argument in the equation, they solved the problem and proved its spectral properties, more on these problem can be found in [46-50].

The authors of [36] gave an interesting result of inverse SLP with a constant delay under a non-self-adjoint operator with a mixed boundary condition, expressing the spectral properties of the eigenvalues obtained and also, proving the uniqueness theorem. Studies and results on fractional inverse SLP are scarce, as such we intended in this paper to express the case in [36] in a fractional case under a conformable derivative operator. We obtained the result of the inverse SLP with a constant delay under a mixed boundary condition using the conformable derivative approach, expressed the corresponding spectral properties, and proved the uniqueness theorem. The corresponding classical results can be retrieved at $\alpha = 1$.

2. Some Basic Definitions

Definition 2.1. Consider the function $h: [0, \infty) \to \mathbb{R}$, then the α^{th} order derivative of h is given

$$D_x^{\alpha} h(x) = \lim_{e \to 0} \frac{h(x + e x^{1-\alpha}) - h(x))}{e} \tag{2.1}$$

for all $x > 0$, $\alpha \in (0,1]$, that is,if h is differentiable, then $D_x^{\alpha} h(x) = x^{1-\alpha} h'(x)$.

The conformable fractional derivative is also defined for $\alpha \in (n-1, n)$ for $n \in N$ as,

Definition 2.2. Let h be an n -differentiable function at x, where $x > 0$ and $\alpha \in (n-1, n)$, then the conformable fractional derivative h of order α is defined as,

$$D_x^{\alpha} h(x) = \lim_{e \to 0} \frac{h^{\lceil \alpha \rceil - 1}(x + e x^{\lceil \alpha \rceil - \alpha}) - h^{\lceil \alpha \rceil - 1}(x)}{e} \tag{2.2}$$

Where $\lceil \alpha \rceil$ is the smallest integer greater than or equal to α.

It can be calculated by $D_x^{\alpha} h(x) = x^{\lceil \alpha \rceil - \alpha} h^{\lceil \alpha \rceil}(x)$

Definition 2.3. The integral of a function h of order α is given by

$$I_{\alpha} h(x) = \int_0^x h(t) d_{\alpha} t = \int_0^x t^{\alpha - 1} h(t) dt \tag{2.3}$$

for all $x > 0$.

Lemma 2.1. If the function $h: [a, \infty) \to \mathbb{R}$ is differentiable, then, we have for $x > a$ (a is any real number)

$$D_x^{\alpha} I_{\alpha} h(x) = h(x).$$

Lemma 2.2. Let the function $h: (a, b) \to \mathbb{R}$ be differentiable, then, for $x > a$,(a and b are any real numbers)

$$D_x^{\alpha} I_{\alpha} h(x) = h(x) - h(a).$$

Theorem 2.4. Let g, h be two differentiable functions, then

$$\int_a^b g(x) D_x^{\alpha}(h(x))(x) d_{\alpha} x = gh|_a^b - \int_a^b h(x) D_x^{\alpha}(g(x)) d_{\alpha} x. \tag{2.4}$$

3. The Main Work

We consider the fractional Sturm-Liouville problem below with conformable derivative operator

$$-D_x^{\alpha} D_x^{\alpha} y + q(x) y(x - a) = \mu y(x), \ for \ \ x \in (0, \pi) \tag{3.1}$$

under the condition

$$y(0) = y^{(j)}(\pi) = 0, \quad for \ j = 0,1, \tag{3.2}$$

for $a \in (0, \pi)$, and $q(x) \in L(a, \pi)$. Taking μ as the spectral parameter and the potential function(complex-valued) $q(x) = 0$ for $x \in [0, a]$.

By defining an operator

$$L_{\alpha} y(x) = -D_x^{\alpha} D_x^{\alpha} y + q(x) y(x - a)$$

then (3.1) can be expressed as

$$L_\alpha y(x) = \mu y(x), \quad x \in (0,\pi) \tag{3.3}$$

The main work here is to recover the function $q(x)$ from the spectra of $L_{\alpha_j}(q),\ j = 0,1.$, to state and prove some properties of the spectra, and also to prove the uniqueness of the results. We assumed that $\{\mu_{n_j}\}_{n\geq 1, j=0,1}$ indicates the eigenvalues of (3.3).

3.1. Existence of the Solution

Consider $N \in N$ such that $a \in \left[\frac{\pi}{N+1}, \frac{\pi}{N}\right]$ and $Q(x,\mu)$ be a solution of (3.3) under the conditions that

$$Q(0,\mu) = 0,\ D_x^\alpha Q(0,\mu) = 1.$$

We can then expressed $Q(x,\mu)$ as

$$Q(x,\mu) = \frac{1}{\sqrt{\mu}} sin\left(\frac{\sqrt{\mu}}{\alpha} x^\alpha\right) + \frac{1}{\sqrt{\mu}} \int_0^x \quad sin\left(\frac{\sqrt{\mu}}{\alpha}(x^\alpha - t^\alpha)\right) q(t) Q(t-a,\mu) d_\alpha t \tag{3.4}$$

clearly, $Q^{(j)}(x,\mu)$, for any x in the interval $(0,\pi)$ and $j = 0,1$, are entire in μ of order $\frac{1}{2}$.

By the method of successive approximations, the solution of (3.4) is

$$Q(x,\mu) = Q_0(x,\mu) + Q_1(x,\mu) + \cdots + Q_N(x,\mu) \tag{3.5}$$

for which,

$$Q_0(x,\mu) = \frac{1}{\sqrt{\mu}} sin\left(\frac{\sqrt{\mu}}{\alpha} x^\alpha\right) \quad for \quad x \geq 0 \tag{3.6}$$

$$Q_k(x,\mu) = \frac{1}{\sqrt{\mu}} \int_{ka}^x \quad sin\left(\frac{\sqrt{\mu}}{\alpha}(x^\alpha - t^\alpha)\right) q(t) Q_{k-1}(t-a,\mu) d_\alpha t \tag{3.7}$$

for $x \geq ka$, and $Q_k(x,\mu) = 0$ for $x \leq ka$.

Now, for $k \geq 1$, and from (3.7) and by Definition 2.1 we have,

$$D_x^\alpha Q_k(x,\mu) = \int_{ka}^x \quad cos\left(\frac{\sqrt{\mu}}{\alpha}(x^\alpha - t^\alpha)\right) q(t) Q_{k-1}(t-a,\mu) d_\alpha t \quad for\ x \geq ka. \tag{3.8}$$

From (3.7) we obtained

$$\begin{aligned} Q_1(x,\mu) &= \frac{1}{\sqrt{\mu}} \int_a^x \quad sin\left(\frac{\sqrt{\mu}}{\alpha}(x^\alpha - t^\alpha)\right) q(t) Q_0(t-a,\mu) d_\alpha t \\ &= \frac{1}{\mu} \int_a^x \quad sin\left(\frac{\sqrt{\mu}}{\alpha}(x^\alpha - t^\alpha)\right). sin\left(\frac{\sqrt{\mu}}{\alpha}(t^\alpha - a^\alpha)\right) q(t) d_\alpha t \end{aligned} \tag{3.9}$$

so that

$$\begin{aligned} Q_1(x,\mu) &= -\frac{1}{2\mu} cos\left(\frac{\sqrt{\mu}}{\alpha}(x^\alpha - a^\alpha)\right) \int_a^x \quad q(t) d_\alpha t \\ &+ \frac{1}{2\mu} \int_a^x \quad cos\left(\frac{\sqrt{\mu}}{\alpha}(2t^\alpha - x^\alpha - a^\alpha)\right) q(t) d_\alpha t \end{aligned} \tag{3.10}$$

then, we have from (3.10) that

$$\begin{aligned} D_x^\alpha(Q_1(x,\mu)) &= \frac{1}{2\sqrt{\mu}} sin\left(\frac{\sqrt{\mu}}{\alpha}(x^\alpha - a^\alpha)\right) \int_a^x \quad q(t) d_\alpha t \\ &+ \frac{1}{2\sqrt{\mu}} \int_a^x \quad sin\left(\frac{\sqrt{\mu}}{\alpha}(2t^\alpha - x^\alpha - a^\alpha)\right) q(t) d_\alpha t. \end{aligned} \tag{3.11}$$

Now, from (3.8) -(3.10), it can be shown that

$$Q_k^{(j)}(x,\mu) = O\left((\sqrt{\mu})^{j-k-1} e^{\left(\frac{1}{\alpha}|Im\sqrt{\mu}|(x^\alpha-(ka)^\alpha)\right)}\right) \tag{3.12}$$

3.2. The Asymptotic Formulae

Let us denote the characteristics function of $L_j(q)$ by $W_j(\mu)$, $j = 0,1$, and $W_j(\mu) = Q^{(j)}(\pi,\mu)$. Since $Q^{(j)}(\pi,\mu)$ are entire in μ of order $\frac{1}{2}$ so also the $W_j(\mu)$.

We drived the asymptotical formulae for the SLP $L_j(q)$ for $|\sqrt{\mu}| \to \infty$ from (3.7), (3.11) and (3.12) as follows,

$$\begin{aligned} W_0(\mu) &= Q(\pi,\mu) = Q_0(\pi,\mu) + Q_1(\pi,\mu) + \cdots + Q_N(\pi,\mu) \\ &= \frac{1}{\sqrt{\mu}} sin\left(\frac{\sqrt{\mu}}{\alpha}\pi^\alpha\right) - \frac{1}{2\mu} cos\left(\frac{\sqrt{\mu}}{\alpha}(\pi^\alpha - a^\alpha)\right)\int_a^\pi \quad q(t)d_\alpha t \\ &\quad + o\left(\mu^{-1} e^{\left(\frac{1}{\alpha}|Im\sqrt{\mu}|(\pi^\alpha-a^\alpha)\right)}\right) \end{aligned} \tag{3.13}$$

so that we have

$$\begin{aligned} W_1(\mu) &= cos\left(\frac{\sqrt{\mu}}{\alpha}\pi^\alpha\right) + \frac{1}{2\sqrt{\mu}} sin\left(\frac{\sqrt{\mu}}{\alpha}(\pi^\alpha - a^\alpha)\right)\int_a^\pi \quad q(t)d_\alpha t \\ &\quad + o\left(\mu^{-\frac{1}{2}} e^{\left(\frac{1}{\alpha}|Im\sqrt{\mu}|(\pi^\alpha-a^\alpha)\right)}\right) \end{aligned} \tag{3.14}$$

The asymptotical formulae for the eigenvalues of the $L_j(q)$ for $\mu_{nj} = \rho_{nj}^2$ as $n \to \infty$ were also obtained using (3.13) and (3.14) and the method described in [37] as

$$\rho_{n_0} = \frac{\alpha n}{\pi^{\alpha-1}} + \frac{1}{2n\pi} cos\left(\frac{n}{\pi^{\alpha-1}} a^\alpha\right)\int_a^\pi \quad q(t)d_\alpha t + O\left(\frac{1}{n}\right) \tag{3.15}$$

and

$$\rho_{n_1} = \frac{\alpha(n-\frac{1}{2})}{\pi^{\alpha-1}} + \frac{1}{2(n-\frac{1}{2})\pi} cos\left(\frac{n}{\pi^{\alpha-1}} a^\alpha\right)\int_a^\pi \quad q(t)d_\alpha t + O\left(\frac{1}{n}\right) \tag{3.16}$$

3.3. The Specification of the Spectrum

The Specification of the Spectrum is one of the spectral properties, as for the spectrum corresponding to a problem for a Sturm-Liouville operator for the interval $(0,\infty)$, it means the complement of the set of points in a neighborhood of which the spectral function $W_j(\mu)$ is constant, as such, to affirm applying the conformable derivative operator, we obtained and proved the specification of the spectrum for the characteristics function as it follows.

Lemma 3.1. The specification of the spectrum $\{\mu_{nj}\}_{n\geq 1}$, $j = 0,1$ uniquely determines the characteristics function $W_j(\mu)$ by the formulas

$$W_0(\mu) = \frac{\pi^{3\alpha-2}}{\alpha^3}\prod_{n=1}^{\infty} \quad \left(\frac{\mu_{n0}-\mu}{n^2}\right) \tag{3.17}$$

and

$$W_1(\mu) = \frac{\pi^{2\alpha-2}}{\alpha^2}\prod_{n=1}^{\infty} \quad \frac{(\mu_{n1}-\mu)}{\left(n-\frac{1}{2}\right)^2} \tag{3.18}$$

Proof. Being $W_j(\mu)$ entire in μ of order $\frac{1}{2}$, then by Hadamard's factorization theorem [51], it can be uniquely determined up to a multiplicative constant by its zeros, that is

$$W_0(\mu) = C\prod_{n=1}^{\infty} \quad \left(1 - \frac{\mu}{\mu_{n0}}\right)$$

Now, since

$$sin\, z = z\prod_{n=1}^{\infty}\left(1-\frac{z^2}{(k\pi)^2}\right)$$

it implies that

$$\widetilde{W}_0(\mu)=\frac{sin\left(\frac{\sqrt{\mu}}{\alpha}\pi^{\alpha}\right)}{\sqrt{\mu}}=\frac{\pi^{\alpha}}{\alpha}\prod_{n=1}^{\infty}\left(1-\frac{\mu\pi^{2\alpha-2}}{\alpha^2n^2}\right)=\frac{\pi^{\alpha}}{\alpha}\prod_{n=1}^{\infty}\left(1-\frac{\mu}{\left(\frac{\alpha^2}{\pi^{2\alpha-2}}\right)n^2}\right)$$

Then,

$$\frac{W_0(\mu)}{\widetilde{W}_0(\mu)}=C\,\frac{\alpha^3}{\pi^{3\alpha-2}}\prod_{n=1}^{\infty}\frac{n^2}{\mu_{n0}}\prod_{n=1}^{\infty}\left(1+\frac{\mu_{n0}-\left(\frac{\alpha^2}{\pi^{2\alpha-2}}\right)n^2}{\left(\frac{\alpha^2}{\pi^{2\alpha-2}}\right)n^2-\mu}\right)$$

Now,

$$\lim_{\mu\to-\infty}\frac{W_0(\mu)}{\widetilde{W}_0(\mu)}=1 \text{ and } \lim_{\mu\to-\infty}\prod_{n=1}^{\infty}\left(1+\frac{\mu_{n0}-\left(\frac{\alpha^2}{\pi^{2\alpha-2}}\right)n^2}{\left(\frac{\alpha^2}{\pi^{2\alpha-2}}\right)n^2-\mu}\right)=1$$

then,

$$C=\frac{\pi^{3\alpha-2}}{\alpha^3}\prod_{n=1}^{\infty}\frac{\mu_{n0}}{n^2}$$

and eventually, we reached

$$W_0(\mu)=C\prod_{n=1}^{\infty}\left(1-\frac{\mu}{\mu_{n0}}\right)=\frac{\pi^{3\alpha-2}}{\alpha^3}\prod_{n=1}^{\infty}\left(\frac{\mu_0-\mu}{n^2}\right).$$

This completes the proof of the first case, that is for $j=0$. The proof for the second case, $j=1$, similarly follows.

3.4. Regge-type Boundary Value Problem

The problem considered in this work, (3.3), can also be extended to Regge-type boundary value problem $L(q)$, that is,

$$y(0)=0,\quad D_x^{\alpha}y(\pi)+i\rho y(\pi)=0.$$

In such a case, the characteristic function will be of the form;

$$S(\mu)=W_1(\mu)+i\rho W_0(\mu) \tag{3.19}$$

which is, also, entire in μ. Now from (3.5) we have

$$S(\mu)=S_0(\mu)+S_1(\mu)+\cdots+S_N(\mu) \tag{3.20}$$

where $S_k(\mu)=D_x^{\alpha}Q_k(\mu)+i\sqrt{\mu}Q_k(\mu)$ which implies that

$$S_0(\mu)=cos\left(\frac{\sqrt{\mu}}{\alpha}\pi^{\alpha}\right)+i\,sin\left(\frac{\sqrt{\mu}}{\alpha}\pi^{\alpha}\right)=e^{i\left(\frac{\sqrt{\mu}}{\alpha}\pi^{\alpha}\right)}$$

From (3.7) and (3.8) we obtained

$$S_k(\mu)=\int_{ka}^{\pi}e^{i\left(\frac{\sqrt{\mu}}{\alpha}(\pi^{\alpha}-t^{\alpha})\right)}q(t)Q_{k-1}(t-a,\mu)d_{\alpha}t,\quad k\geq 1 \tag{3.21}$$

It follows from (3.10) and (3.11) that

$$S_1(\mu) = \frac{1}{2i\sqrt{\mu}}\left(e^{\left(\frac{i\sqrt{\mu}}{\alpha}(\pi^\alpha - a^\alpha)\right)}\int_a^\pi \quad q(t)d_\alpha t - e^{\left(\frac{i\sqrt{\mu}}{\alpha}(\pi^\alpha + a^\alpha)\right)}\int_a^\pi \quad q(t)e^{\left(\frac{-2i\sqrt{\mu}}{\alpha}t^\alpha\right)}d_\alpha t\right) \tag{3.22}$$

by virtue of (3.12) and (3.21) we obtain

$$S_k(\mu) = O\left((\sqrt{\mu})^{-k}\int_{ka}^\pi \quad q(t)e^{\left(\frac{i\sqrt{\mu}}{\alpha}(2t^\alpha - \pi^\alpha - (ka)^\alpha)\right)}d_\alpha t\right), \tag{3.23}$$

for $Im\sqrt{\mu} \geq 0, \quad |\sqrt{\mu}| \to \infty. \quad k \geq 1.$

4. Uniqueness Theorem

In this part of the study, we want to give and prove the uniqueness theorems, which show that the potential functions of the two different problems are the same if the spectrums are coincident under different conditions. It should be noted that we have expressed our problem under a different set of boundary conditions; the mixed and the Regge-type, and we will use them to prove the uniqueness. These theorems were given for the classical derivative problems, however, we aim to extend this derivative to the entire real interval [0,1] and bring the results to the literature. We must state that in the case where $\alpha = 1$, the results coincided with the results given in [36].

To prove the uniqueness theorem, let $\{\tilde{\mu}_{nj}\}_{n\geq 1}, j = 0,1$, be the eigenvalues of the problems $\tilde{L}_j = L_j(\tilde{q})$ with the potential $\tilde{q}(x) = 0$, then $\tilde{\mu}_{n0} = (\frac{\alpha n}{\pi^{\alpha-1}})^2$ and $\tilde{\mu}_{n1} = (\frac{\alpha(n-\frac{1}{2})}{\pi^{\alpha-1}})^2$, for $n \geq 1$.

Consider $\tilde{S}(\mu)$ as the characteristic function of $\tilde{L} = L(\tilde{q})$. It implies from (3.19) that $\tilde{S}(\mu) = e^{\frac{i\sqrt{\mu}}{\alpha}\pi^\alpha}$.

Theorem 4.1. If $\mu_{nj} = \tilde{\mu}_{nj}, \forall \quad n \geq 1, \ for \ j = 0,1$, then the potential $q(x) = 0$ almost everywhere on (a, π).

Proof. From lemma 3.1 and the special infinite series, we have

$$W_0(\mu) = \frac{sin\left(\frac{\sqrt{\mu}}{\alpha}\pi^\alpha\right)}{\sqrt{\mu}} \text{ and } W_1(\mu) = cos\left(\frac{\sqrt{\mu}}{\alpha}\pi^\alpha\right)$$

as such, $S(\mu) = e^{\frac{i\sqrt{\mu}}{\alpha}\pi^\alpha}$. From (3.20), we can deduce that

$$S_1(\mu) = -S^+(\mu) \tag{4.1}$$

where, $S^+(\mu) = \sum_{k=2}^N \quad S_k(\mu)$, $k \geq 2$ with $S^+(\mu) = 0$, $k = 1$.

Taking $\mu_{nj} = \tilde{\mu}_{nj}$, (3.15) and (3.16) implies that $\int_a^\pi \quad q(t)d_\alpha t = 0$, then (3.22) yields

$$S_1(\mu) = -\frac{1}{2i\sqrt{\mu}}e^{\left(\frac{i\sqrt{\mu}}{\alpha}(\pi^\alpha + a^\alpha)\right)}\int_a^\pi \quad q(t)e^{\left(\frac{-2i\sqrt{\mu}}{\alpha}t^\alpha\right)}d_\alpha t \tag{4.2}$$

Let $N = 1$, i.e. $a \in [\frac{\pi}{2}, \pi]$, then $S^+(\mu) = 0$ which implies from(4.1) that $S_1(\mu) = 0$ as such, (4.2) gives,

$$\int_a^\pi \quad q(t)e^{\left(\frac{-2i\sqrt{\mu}}{\alpha}t^\alpha\right)}d_\alpha t = 0$$

and the only possibility is $q(x) = 0$ almost everywhere on (a, π). This completes the proof for $N = 1$ and below is for $N \geq 2$.

Lemma 4.1. If the potential $q(x) = 0$ almost everywhere on $(2a, \pi)$, then $q(x) = 0$ almost everywhere on (a, π).

Proof. Let $q(x) = 0$ almost everywhere on $(2a, \pi)$, from (3.21) $S_k(\mu) = 0$ for $k \geq 2$ and hence $S^+(\mu) = 0$, then from (4.1) we have $S_1(\mu) = 0$ and consequently $q(x) = 0$ almost everywhere on (a, π). This completes the proof of lemma 4.1.

To make it more clear, let's consider the N in two ways; odd and even. Firstly, we will assume that $N = 2M + 1$, i.e. N is odd, in the following.

Lemma 4.2. Let $d = 0,1,3, \dots 2M - 1$. If the potential $q(x) = 0$ almost everywhere on $\left(\pi - \frac{da}{2}, \pi\right)$, then $q(x) = 0$ almost everywhere on $\left(\pi - \frac{(d+1)a}{2}, \pi\right)$.

Proof. From the fact that $\left(\pi - \frac{da}{2}, \pi\right) > 2a$, it follows (3.23) that

$$S_2(\mu) = O\left(\frac{1}{\mu}\int_{2a}^{\pi-\frac{da}{2}} q(t)e^{\left(\frac{i\sqrt{\mu}}{\alpha}(2t^\alpha - \pi^\alpha - (2a)^\alpha)\right)}d_\alpha t\right),\ Im\sqrt{\mu} \geq 0,\ |\sqrt{\mu}| \to \infty$$

clearly, $2t - \pi - 2a \in (2a - \pi, \pi - (d+2)a)$, for $\pi - (d+2)a) \geq \pi - Na$ which yields,

$$S_2(\mu) = O\left(\frac{1}{\mu}e^{\left(\frac{-i\sqrt{\mu}}{\alpha}(\pi^\alpha - ((d+2)a)^\alpha)\right)}d_\alpha t\right),\ Im\sqrt{\mu} \geq 0,\ |\sqrt{\mu}| \to \infty. \tag{4.3}$$

$S_k(\mu)$ in the equation (4.3), increase less rapidly than the right-hand side when $k \geq 2$, that is,

$$S^+(\mu) = O\left(\frac{1}{\mu}e^{\left(\frac{-i\sqrt{\mu}}{\alpha}(\pi^\alpha - ((d+2)a)^\alpha)\right)}d_\alpha t\right),\ Im\sqrt{\mu} \geq 0,\ |\sqrt{\mu}| \to \infty \tag{4.4}$$

it follows from (4.1), (4.2) and (4.4) that

$$e^{\left(\frac{i\sqrt{\mu}}{\alpha}(\pi^\alpha + a^\alpha)\right)}\int_a^{\pi-\frac{da}{2}} q(t)e^{\left(\frac{-2i\sqrt{\mu}}{\alpha}t^\alpha\right)}d_\alpha t = O\left(\frac{1}{\sqrt{\mu}}e^{\left(\frac{-i\sqrt{\mu}}{\alpha}(\pi^\alpha - ((d+2)a)^\alpha)\right)}\right),$$

$Im\sqrt{\mu} \geq 0,\ |\sqrt{\mu}| \to \infty,$

which can be expressed as

$$e^{\left(\frac{i\sqrt{\mu}}{\alpha}(2\pi^\alpha + (1-(d+2)^\alpha)a^\alpha)\right)}\int_a^{\pi-\frac{da}{2}} q(t)e^{\left(\frac{-2i\sqrt{\mu}}{\alpha}t^\alpha\right)}d_\alpha t = O\left(\frac{1}{\sqrt{\mu}}\right), \tag{4.5}$$

$Im\sqrt{\mu} \geq 0, \quad |\sqrt{\mu}| \to \infty.$

Furthermore, we have

$$\int_a^{\pi-\frac{(d+1)a}{2}} q(t)e^{\left(\frac{-2i\sqrt{\mu}}{\alpha}t^\alpha\right)}d_\alpha t = O\left(e^{\left(\frac{-i\sqrt{\mu}}{\alpha}(2\pi^\alpha + (1-(d+2)^\alpha)a^\alpha\right)}d_\alpha t\right), \tag{4.6}$$

$Im\sqrt{\mu} \geq 0, \quad |\sqrt{\mu}| \to \infty.$

Now, let's define the function,

$$G(\sqrt{\mu}) = e^{\left(\frac{i\sqrt{\mu}}{\alpha}(2\pi^\alpha + (1-(d+2)^\alpha)a^\alpha)\right)}\int_{\pi-\frac{(d+1)a}{2}}^{\pi-\frac{da}{2}} q(t)e^{\left(\frac{-2i\sqrt{\mu}}{\alpha}t^\alpha\right)}d_\alpha t \tag{4.7}$$

which is entirely in μ. Clearly, $G(\sqrt{\mu}) = O(1)$ for $Im\sqrt{\mu} \leq 0$, also, it follows from (4.5) and (4.6) that $G(\sqrt{\mu}) = O(1)$ for $Im\sqrt{\mu} \geq 0$. Since the function $G(\sqrt{\mu})$ is entire bounded it follows from Liouville's theorem [51] that $G(\sqrt{\mu}) = c$, where c is a constant. From $G(\sqrt{\mu}) = o(1)$ for real $\sqrt{\mu}$, $|\sqrt{\mu}| \to \infty$, then $G(\sqrt{\mu}) = 0$, hence (4.7) gives

$$\int_{\pi-\frac{(d+1)a}{2}}^{\pi-\frac{va}{2}} q(t)e^{\left(\frac{-2i\sqrt{\mu}}{\alpha}t^\alpha\right)}d_\alpha t = 0$$

which gives $q(x) = 0$ almost everywhere on $\left(\pi - \frac{(d+1)a}{2}, \pi - \frac{da}{2}\right)$ which completes the proof.

We obtained $q(x) = 0$ almost everywhere on $(\pi - Ma, \pi)$ by applying lemma 4.2 successively for $d = 0,1,3, \dots 2m -$

1.

We noted that lemma 4.2 is for the odd case. Now, let $d \geq 2M$ such that N is even.

Lemma 4.3. If the potential $q(x) = 0$ almost everywhere on $\pi - Ma, \pi$, then $q(x) = 0$ almost everywhere on $\left(\frac{(M+2)a}{2}, \pi\right)$.

Proof. If $k \geq M + 2$, we then have $\pi - Ma - ka \leq \pi - (N + 1)a \leq 0$ and hence, $S_\mu = 0$ for $k \geq M + 2$.

According to (3.23), for $k = 2,3,4, \dots, M + 1$,

$$S_k(\mu) = O\left((\sqrt{\mu})^{-k} \int_{ka}^{\pi - Ma} q(t) e^{\left(\frac{-i\sqrt{\mu}}{\alpha}(2t^\alpha - \pi^\alpha - (ka)^\alpha)\right)} d_\alpha t\right), \tag{4.8}$$

for $Im\sqrt{\mu} \geq 0, \quad |\sqrt{\mu}| \to \infty$.

Being $2t - \pi - ka \leq 0$, it follows that

$$S_k(\mu) = O\left((\sqrt{\mu})^{-k} e^{\left(\frac{i\sqrt{\mu}}{\alpha}(\pi^\alpha - (ka)^\alpha)\right)}\right), \tag{4.9}$$

for $Im\sqrt{\mu} \geq 0, \quad |\sqrt{\mu}| \to \infty, \ for \ k = 2,3,4, \dots, M + 1$ and hence

$$S^+(\mu) = O\left(\frac{1}{\mu} e^{\left(\frac{i\sqrt{\mu}}{\alpha}(\pi^\alpha - ((M+1)a)^\alpha)\right)}\right), \tag{4.10}$$

for $Im\sqrt{\mu} \geq 0, \quad |\sqrt{\mu}| \to \infty$,

As a result of (4.1), (4.2) and (4.9) we obtained

$$e^{\left(\frac{i\sqrt{\mu}}{\alpha}(\pi^\alpha + a^\alpha)\right)} \int_a^{\pi - Ma} q(t) e^{\left(\frac{-2i\sqrt{\mu}}{\alpha} t^\alpha\right)} d_\alpha t = O\left(\frac{1}{\sqrt{\mu}} e^{\left(\frac{i\sqrt{\mu}}{\alpha}(\pi^\alpha - ((M+1)a)^\alpha)\right)}\right),$$

$Im\sqrt{\mu} \geq 0, \ |\sqrt{\mu}| \to \infty$ or, which is equivalent to,

$$e^{\left(\frac{i\sqrt{\mu}}{\alpha}((M+2)a)^\alpha\right)} \int_a^{\pi - Ma} q(t) e^{\left(\frac{-2i\sqrt{\mu}}{\alpha} t^\alpha\right)} d_\alpha t = O\left(\frac{1}{\sqrt{\mu}}\right), \ Im\sqrt{\mu} \geq 0, \ |\sqrt{\mu}| \to \infty \tag{4.11}$$

furthermore,

$$\int_a^{\frac{(M+2)a}{2}} q(t) e^{\left(\frac{-2i\sqrt{\mu}}{\alpha} t^\alpha\right)} d_\alpha t = O\left(e^{\left(\frac{-i\sqrt{\mu}}{\alpha}((M+2)a)^\alpha\right)}\right) \tag{4.12}$$

$Im\sqrt{\mu} \geq 0, \ |\sqrt{\mu}| \to \infty$

Let us denote

$$G^*(\sqrt{\mu}) = e^{\left(\frac{i\sqrt{\mu}}{\alpha}((M+2)a)^\alpha\right)} \int_{\frac{(M+2)a}{2}}^{\pi - Ma} q(t) e^{\left(\frac{-2i\sqrt{\mu}}{\alpha} t^\alpha\right)} d_\alpha t$$

Which is entire in μ and $G^*(\sqrt{\mu}) = O(1)$ for $Im\sqrt{\mu} \leq 0$. In view of (4.11) and (4.12), $G^*(\sqrt{\mu}) = O(1)$ for $Im\sqrt{\mu} \geq 0$. Therefore,as in the above similar case, $G^*(\sqrt{\mu}) = C$, since $G^*(\sqrt{\mu}) = o(1)$ for real $\sqrt{\mu}$, $|\sqrt{\mu}| \to \infty$, then $G^*(\sqrt{\mu}) = 0$, that is,

$$\int_{\frac{(M+2)a}{2}}^{\pi - \frac{Ma}{2}} q(t) e^{\left(\frac{-2i\sqrt{\mu}}{\alpha} t^\alpha\right)} d_\alpha t = 0.$$

This implies that $q(x) = 0$ almost everywhere on $\left(\frac{(M+2)a}{2}, \pi - Ma\right)$ which completes the proof.

It has been proved that $q(x) = 0$ almost everywhere on $(2a, \pi)$ for $M = 1$ or $M = 2$. According to lemma 4.1, $(x) = 0$ almost everywhere on (a, π). Therefore, theorem 4.1 is proved for $M = 1$ and $M = 2$.

Let now $M \geq 3$. Fix $d = 5,6,7,8, \ldots, M + 2$. Let $l = \frac{(d+1)}{2}$. Clearly, $l < d$.

Lemma 4.4. If the potential $q(x) = 0$ almost everywhere on $\left(\frac{da}{2}, \pi\right)$, then $q(x) = 0$ almost everywhere on $\left(\frac{la}{2}, \pi\right)$.

Proof. Considering $\frac{d}{2} - k \leq \frac{d}{2} - l \leq 0$ for $k \geq l$, we have $S_k(\mu) = 0$ for $k \geq l$. By virtue of (3.23)

$$S_k(\mu) = O\left((\sqrt{\mu})^{-k} \int_{ka}^{\frac{da}{2}} q(t) e^{\left(\frac{-i\sqrt{\mu}}{\alpha}(2t^\alpha - \pi^\alpha - (ka)^\alpha)\right)} d_\alpha t\right), \tag{4.13}$$

for $Im\sqrt{\mu} \geq 0, \quad |\sqrt{\mu}| \to \infty, \quad k = 2,3,4, \ldots, l-1$. Now, $2t - \pi - ka < 0$ that is, the exponent is decreasing for $Im\sqrt{\mu} > 0$, then,

$$S_k(\mu) = O\left((\sqrt{\mu})^{-k} e^{\left(\frac{i\sqrt{\mu}}{\alpha}(\pi^\alpha - (ka)^\alpha)\right)}\right), \tag{4.14}$$

for $Im\sqrt{\mu} \geq 0, \quad |\sqrt{\mu}| \to \infty, \quad k = 2,3,4, \ldots, l-1$, therefore,

$$S^+(\mu) = O\left(\frac{1}{\mu} e^{\left(\frac{i\sqrt{\mu}}{\alpha}(\pi^\alpha - ((l-1)a)^\alpha)\right)}\right), \; Im\sqrt{\mu} \geq 0, \quad |\sqrt{\mu}| \to \infty, \tag{4.15}$$

from (4.1), (4.2) and (4.15) we obtained

$$e^{\left(\frac{i\sqrt{\mu}}{\alpha}(la)^\alpha\right)} \int_0^{\frac{da}{2}} q(t) e^{\left(\frac{-2i\sqrt{\mu}}{\alpha} t^\alpha\right)} d_\alpha t = O\left(\frac{1}{\sqrt{\mu}}\right) \tag{4.16}$$

for $Im\sqrt{\mu} \geq 0, \quad |\sqrt{\mu}| \to \infty$. moreover,

$$\int_a^{\frac{la}{2}} q(t) e^{\left(\frac{-2i\sqrt{\mu}}{\alpha} t^\alpha\right)} d_\alpha t = O\left(e^{\left(\frac{-i\sqrt{\mu}}{\alpha}(la)^\alpha\right)}\right) \; for \; Im\sqrt{\mu} \geq 0, \; |\sqrt{\mu}| \to \infty. \tag{4.17}$$

$Im\sqrt{\mu} \geq 0, \; |\sqrt{\mu}| \to \infty$ Let us denote,

$$G^{**}(\sqrt{\mu}) = e^{\left(\frac{i\sqrt{\mu}}{\alpha}((la)^\alpha)\right)} \int_{\frac{(la)}{2}}^{\frac{da}{2}} q(t) e^{\left(\frac{-2i\sqrt{\mu}}{\alpha} t^\alpha\right)} d_\alpha t$$

Which is entire in μ as well, and $G^{**}(\sqrt{\mu}) = O(1)$ for $Im\sqrt{\mu} \leq 0$. Considering (4.16) and (4.17), $G^{**}(\sqrt{\mu}) = O(1)$ for $Im\sqrt{\mu} \geq 0$. consequently $G^{**}(\sqrt{\mu}) = o(1)$ for real $\sqrt{\mu}, \; |\sqrt{\mu}| \to \infty$, therefore $G^{**}(\sqrt{\mu}) = 0$ and as a result of which $q(x) = 0$ almost everywhere on $\left(\frac{la}{2}, \frac{da}{2}\right)$. Hence lemma 4.4 is proved.

Applying the lemma 4.4 many times consecutively starting from $d = M + 2$, we obtain that the potential $q(x) = 0$ almost everywhere on $(2a, \pi)$. Therefore, due to lemma 4.1, $q(x) = 0$ almost everywhere on (a, π). Hence, this completes the proof of the theorem.

It can be seen that, with regard to our problem we proved the existence of the solution, the spectral properties, and also the uniqueness theorem in detail using the proposed fractional approach and these completes our work.

5. Conclusion

In conclusion, the method of conformable derivative, which is more accessible to the other existing fractional derivative approaches due to its satisfying properties, has been used in this work as a derivative operator with which we show and express the possibility of solving the inverse SLP with constant delay. The problem discussed is under mixed boundary conditions and in each case, a result is obtained. Also, the specifications of the respective spectrums are given affirming the solution obtained. The asymptotics of the eigenvalues were extended to the Regge-type boundary value problem and analyzed. The proof of the uniqueness theorem is similar to the one in the classical case of the problem. Similar problems with different boundary conditions can be discussed under this new fractional derivative approach with their corresponding spectral properties and this will lead to providing an entire phase of fractional SLP.

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Conflict of Interest

The authors have no competing interests to declare that are relevant to the content of this article.

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