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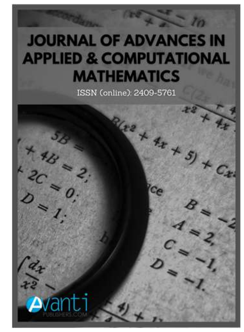

A Four Step Scheme Approach to the Forward-Backward Stochastic Navier-Stokes Equations

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ABSTRACT

In this paper, the authors presented a novel fluid dynamics system, the forward-backward stochastic Navier-Stokes equations in two dimensions for incompressible fluid flows. The well-posedness of the system is obtained through a two-step process. First, certain projections of the system to the finite dimensions are employed, and the existence and uniqueness of solutions in finite dimensions are proved via the four step scheme. Then the Galerkin approximation is used to show the existence and uniqueness of a solution to the system in an infinite dimensional functional setup.

2000 *Mathematics Subject Classification*. Primary 60H15, 35Q30; Secondary 76D05, 35R60 backward stochastic differential equations.

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1. Introduction

Linear backward stochastic differential equations were introduced by Bismut in 1973 [2], and the systematic study of general backward stochastic differential equations (BSDEs, for short) was put forward first by Pardoux and Peng in 1990 [14]. Since the theory of BSDEs is well connected with nonlinear partial differential equations, nonlinear semigroups, and stochastic controls, it has been intensively studied in the past two decades. The motivation of forward-backward stochastic differential equations (FBSDEs, for short) comes from stochastic control theory. It was initiated by Bismut [3] and expanded by Bensoussan [1], Peng [15], and others. The four step scheme was introduced by Ma, Protter, and Yong [10] in 1994. This effective method provides explicit relations among the forward and backward components of the adapted solution via a quasilinear partial differential equation. The method of continuation was introduced by Hu-Peng [6], Peng-Wu [16] and Yong [23]. This method allows non-Markovian structure, but requires the "monotonicity" conditions on the coefficients. An equivalence relationship between the well-posedness of FBSDEs with random coefficients and that of *backward stochastic PDEs* (BSPDEs) was studied by Ma, Yin, and Zhang [11]. Using either the method of contraction mapping or the method of continuity, forward-backward stochastic partial differential equations (FBSPDEs) were first studied by Yin [20]. Furthermore, contrary to the common belief, Yin [21] removed the monotonicity assumption in the solvability of FBSPDEs. In a more recent work [22], with the help of the Yosida approximation scheme, Yin showed the solvability of a class of fully-coupled FBSPDEs could be achieved without the assumption of Lipschitz conditions.

To characterize the motion of fluids mathematically, a mass of work was done in the 18th and 19th centuries. In 1755, Euler proposed the Euler equations for fluids in the absence of viscosity. The ubiquitous dissipation that is due to the friction of one parcel of fluid against neighboring ones was not well represented, and the appropriate viscous transport was introduced to the Euler equations by Navier and Stokes in 1822 and 1845, which resulted in the now-famous Navier-Stokes equation. Backward stochastic Navier-Stokes equations (BSNSEs) were first introduced by Sundar and Yin [18]. A backward stochastic Navier-Stokes system can be viewed as an inverse problem wherein the velocity profile at a time T is observed and given, and the noise coefficient has to be ascertained from the given terminal data. Such a motivation arises naturally when one understands the importance of inverse problems in partial differential equations (see J. L. Lions [8, 9]). In [18], the authors proved the well-posedness of BSNSEs in two dimensions using a monotonicity argument. The results were further advanced [17] to three dimensions through a series of finite-dimensional projections, linearized equations, and suitable truncations. In this paper, a new version of stochastic Navier-Stokes equations (SNSE for short), namely the forward-backward stochastic Navier-Stokes equations (FBSNSEs for short), are introduced and studied. This type of equation can be used in fluid dynamics to describe situations such as the confluence of two streams. For instance, suppose stream A and stream B with velocity $\mathbf{u}$ and $\mathbf{v}$, respectively, are confluent somewhere within the time interval $(0,T)$. Suppose we know the initial velocity of stream A, i.e., $\mathbf{u}(0)$, and the relationship between $\mathbf{u}$ and $\mathbf{v}$ at time T, namely $\mathbf{v}(T) = g(\mathbf{u}(T))$ for some function g. Let us consider the following system

$$\begin{cases} d\mathbf{u}(t) = -\nu\mathbf{A}\mathbf{u}(t)dt - \mathbf{B}(\mathbf{u}(t))dt + \mathbf{f}(t)dt + \sigma(t,\mathbf{v}(t))dW(t) \\ d\mathbf{v}(t) = \mu\mathbf{A}\mathbf{v}(t)dt + \mathbf{B}(\mathbf{v}(t))dt + \mathbf{g}(t)dt + Z(t)dW(t) \\ \mathbf{u}(0,x) = \mathbf{u}_0(x), \text{ and } \mathbf{v}(T) = g(\mathbf{u}(T)), t \in [0,T],\ \mathbf{x} \in G, \end{cases}$$

and leave the explanation of the coefficients to the next section. The question is can we find the velocities $\mathbf{u}$ and $\mathbf{v}$? This kind of approach of modeling stream confluence is new, to our best knowledge.

The structure of this paper is as follows. In section 2, we introduce some background knowledge and formulate the problem. We studied the backward stochastic Navier-Stokes equation in section 3 and proved the well-posedness of the backward system. In order to understand the infinite-dimensional problem, a finite-dimensional projection is introduced in section 4. Some a priori estimates of finite-dimension are also given in this section. In section 5, we project the system to finite-dimensional real spaces and employ the four-step scheme. Finally, the Galerkin approximation is applied in section 6 to solve the equations in the original functional spaces.

2. Background and Formulation

Let $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},P)$ be a stochastic basis, where P is a probability measure, $\mathcal{F}$ is a σ -algebra and $\{\mathcal{F}_t\}_{t\geq 0}$ is a right-continuous filtration such that all P-null sets are in $\mathcal{F}_0$. Let $W(t)$ be Hilbert-valued Wiener process with covariance operator Q. Let G be a bounded domain in $\mathbb{R}^2$ with smooth boundary ∂G. The Stochastic Navier-Stokes equation is given by

$$\begin{cases} \partial \mathbf{u} + \{(\mathbf{u}\cdot\nabla)\mathbf{u} - \nu\Delta\mathbf{u}\}dt = \{-\nabla p + \mathbf{f}(t)\}dt + \sigma(t)dW(t) \\ \nabla\cdot\mathbf{u} = 0 \\ \mathbf{u}(0,x) = \mathbf{u}_0(x),\ t\in[0,T],\ \mathbf{x}\in G, \end{cases} \tag{2.1}$$

where $\mathbf{u}$ is the velocity profile, ν is the viscosity constant, p is the scalar-valued pressure field, $\mathbf{f}$ is the external body force and σ is the random noise term. Let P denote the Leray-Hodge projection $P: (L^2(G))^2 = \mathrm{H}+\mathrm{H}^{\perp}\rightarrow\mathrm{H}$, where

$$\mathrm{H}^{\perp} \triangleq \{\mathbf{g}\in(L^2(G))^2 : \mathbf{g} = \nabla h \text{ where } h\in W^{1,2}(G)\},$$

and

$$\mathrm{H} \triangleq \{\mathbf{u}\in(L^2(G))^2 : \text{div}(\mathbf{u}) = \nabla\cdot\mathbf{u} = 0 \text{ and } \gamma(\mathbf{u}) = \mathbf{u}\cdot\mathbf{n}_G = 0\},$$

the collection of all vector fields which are square-integrable, divergence free, and tangent to the boundary. Define a dense subspace of H as follows:

$$\mathrm{V} \triangleq \{\mathbf{u}\in(H^1(G))^2 : \nabla\cdot\mathbf{u} = 0 \text{ and} \gamma_0(\mathbf{u}) = \mathbf{u}|_{\partial G} = 0\}.$$

Clearly, they are separable Hilbert spaces, and the embedding is continuous and compact. Denote $(\cdot,\cdot)$the inner product of H, $(\cdot,\cdot)_{\mathrm{V}}$ the inner product of V=$\mathrm{V}(G)$, V' the dual space of V, and $\langle\cdot,\cdot\rangle$ the duality. Let $|\cdot|$ be the norm of H and $\|\cdot\|$ be the norm of V. They are given as follows: $|\mathbf{u}| \triangleq (\int_G |\mathbf{u}|^2 dx)^{\frac{1}{2}}$, and $\|\mathbf{u}\| \triangleq (\int_G |\nabla\mathbf{u}|^2 dx)^{\frac{1}{2}}$. For notational simplicity, the norm $|\cdot|$ inside the integral signs is also used to denote the standard norm on $\mathbb{R}^n$, $n\in\mathbb{N}$. For any $\mathbf{x}\in\mathrm{H}$ and $\mathbf{y}\in\mathrm{V}$, there exists $\mathbf{x}'\in\mathrm{V}'$, such that $(\mathbf{x},\mathbf{y}) = \langle\mathbf{x}',\mathbf{y}\rangle$. Then the mapping $\mathbf{x}\mapsto\mathbf{x}'$ is linear, injective, compact, and continuous. We identify H' with H, and

$$\mathrm{V}\subset\mathrm{H}=\mathrm{H}'\subset\mathrm{V}'$$

is a Gelfand triple. Under the orthogonal projection P, we can write (2.1) in the analytical form as

$$\begin{cases} d\mathbf{u}(t) = -\nu\mathbf{A}\mathbf{u}(t)dt - \mathbf{B}(\mathbf{u}(t))dt + \mathbf{f}(t)dt + \sigma(t)dW(t) \\ \mathbf{u}(0,\mathbf{x}) = \mathbf{u}_0(\mathbf{x}),\ t\in[0,T],\ \mathbf{x}\in G. \end{cases} \tag{2.2}$$

The pressure term p vanishes since ∇p belongs to $\mathrm{H}^{\perp}$. The stokes operator $\mathbf{A}$ is defined as $\mathbf{A}\mathbf{u} \triangleq -\mathrm{P}(\Delta u)$, and it is linear and unbounded. The nonlinear term $\mathbf{B}(\mathbf{u},\mathbf{v}) \triangleq \mathrm{P}((\mathbf{u}\cdot\nabla)\mathbf{v})$ can be abbreviated as $\mathbf{B}(\mathbf{u}) \triangleq \mathbf{B}(\mathbf{u},\mathbf{u})$, and $b(\mathbf{u},\mathbf{v},\mathbf{w}) \triangleq \langle\mathbf{B}(\mathbf{u},\mathbf{v}),\mathbf{w}\rangle$ is a trilinear continuous form on $\mathrm{V}\times\mathrm{V}\times\mathrm{V}$. It can be shown that $b(\mathbf{u},\mathbf{v},\mathbf{w}) = -b(\mathbf{u},\mathbf{w},\mathbf{v})$. Let L_Q denote the space of all linear operators E such that $EQ^{\frac{1}{2}}$ is a Hilbert-Schmidt operator from H to H, with the inner product $\langle F,G\rangle_{L_Q} = tr(FQG^*) = tr(GQF^*)$ for all F and $G\in L_Q$. Let $\{\mathbf{e}_i\}_{i=1}^{\infty}$ be a complete orthonormal system of H such that there exists an increasing family of positive numbers $\{\lambda_i\}_{i=1}^{\infty}$ with limit being ∞, $\mathbf{A}\mathbf{e}_i$=$\lambda_i\mathbf{e}_i$ for all $i\in\mathbb{N}$. Let $q_i = Q\mathbf{e}_i$ for all i. Note that Q is self-adjoint and positive definite nuclear operator. Hence $\sum_{i=1}^{\infty} q_i < \infty$. The H –valued Q -Wiener process W can be defined as $W(t) \triangleq \sum_{i=1}^{\infty}\sqrt{q_i}b^i(t)\mathbf{e}_i$, where $\{b^i(t)\}$ is a sequence of i.i.d. Brownian motions in $\mathbb{R}$. The reader may refer to [4] for more details on Hilbert space-valued Wiener processes.

Now let us consider the time-reversed Navier-Stokes problem. In the deterministic forward equation, replacing t by $T-t$, one gets

$$\begin{cases} -d\mathbf{u}(T-t) = -\nu\mathbf{A}\mathbf{u}(T-t)dt - \mathbf{B}(\mathbf{u}(T-t))dt + \mathbf{f}(T-t)dt \\ \mathbf{u}(0) = \mathbf{u}_0,\ t\in[0,T]. \end{cases}$$

Let $\mathbf{v}(t) = \mathbf{u}(T-t)$ and $\mathbf{g}(t) = -\mathbf{f}(T-t)$. The above equation becomes

$$\begin{cases} d\mathbf{v}(t) = \nu \mathbf{A}\mathbf{v}(t)dt + \mathbf{B}(\mathbf{v}(t))dt + \mathbf{g}(t)dt \\ \mathbf{v}(T) = \mathbf{u}_0, \ t \in [0,T]. \end{cases}$$

Based on the above reasoning and under the usual setup, we consider the following forward-backward stochastic Navier-Stokes equation in this paper:

$$\begin{cases} d\mathbf{u}(t) = -\nu \mathbf{A}\mathbf{u}(t)dt - \mathbf{B}(\mathbf{u}(t))dt + \mathbf{f}(t)dt + \sigma(t,\mathbf{v}(t))dW(t) \\ d\mathbf{v}(t) = \mu \mathbf{A}\mathbf{v}(t)dt + \mathbf{B}(\mathbf{v}(t))dt + \mathbf{g}(t)dt + Z(t)dW(t) \\ \mathbf{u}(0,\mathbf{x}) = \mathbf{u}_0(\mathbf{x}), \text{ and } \mathbf{v}(T) = \mathrm{g}(\mathbf{u}(T)), \ t \in [0,T], \ \mathbf{x} \in G. \end{cases} \tag{2.3}$$

The solution is a triple $(\mathbf{u}, \mathbf{v}, Z)$, where $Z \in L_Q$.

Definition 2.1. A triple of processes $(\mathbf{u}, \mathbf{v}, Z)$ is called a solution of (2.3) if each component is $\mathcal{F}_t$-adapted, square-integrable and satisfies (2.3) P-almost surely.

The following simple results are frequently used and given as lemmas. The reader may refer to Temam[19] for similar proofs.

Lemma 2.2. For any $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w} \in \mathbb{V}$, we have

1. $\langle \mathbf{A}\mathbf{u}, \mathbf{w}\rangle = \sum_{i,j} \int_G \partial_i u_j \, \partial_i w_j dx = \langle \mathbf{A}\mathbf{w}, \mathbf{u}\rangle = (\mathbf{u}, \mathbf{w})_{\mathbb{V}}$,
2. $\langle (\mathbf{u} \cdot \nabla)\mathbf{v}, \mathbf{w}\rangle = \sum_{i,j} \int_G u_i (\partial_i v_j) w_j dx$,
3. $\langle (\mathbf{u} \cdot \nabla)\mathbf{v}, \mathbf{w}\rangle = -\langle (\nabla \cdot \mathbf{u})\mathbf{w}, \mathbf{v}\rangle - \langle (\mathbf{u} \cdot \nabla)\mathbf{w}, \mathbf{v}\rangle$,
4. $\langle \mathbf{B}(\mathbf{u}, \mathbf{v}), \mathbf{w}\rangle = -\langle \mathbf{B}(\mathbf{u}, \mathbf{w}), \mathbf{v}\rangle$.

Corollary 2.3. For any $\mathbf{u}$ and $\mathbf{v} \in \mathbb{V}$, we have

1. $\|\mathbf{u}\| = |\mathbf{A}^{\frac{1}{2}}\mathbf{u}|$,
2. $\langle \mathbf{B}(\mathbf{u}, \mathbf{v}), \mathbf{v}\rangle = 0$.

Lemma 2.4. The following results hold for any real-valued smooth functions φ and ψ with compact support in $\mathbb{R}^2$:

$$|\varphi\psi|^2 \le C\|\varphi\,\partial_1\varphi\|_{\mathrm{L}^1}\|\psi\,\partial_2\psi\|_{\mathrm{L}^1},$$

$$\|\varphi\|_{\mathrm{L}^4}^4 \le C|\varphi|^2|\nabla\varphi|^2.$$

Proposition 2.5. [5] Let $G \subset \mathbb{R}^n$ be bounded, open and of class C^l where $l \ge 1$. Then there exists a constant $C_G > 0$, a scale-invariant constant, such that

$$|\langle \mathbf{B}(\mathbf{u}, \mathbf{v}), \mathbf{w}\rangle| \le C_G |\mathbf{u}|^{1/2}\|\mathbf{u}\|^{1/2}\|\mathbf{v}\||\mathbf{w}|^{1/2}\|\mathbf{w}\|^{1/2}$$

for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{V}$.

3. The Backward System with L^2-Integrable Terminal Conditions

Let us first discuss the following backward stochastic Navier-Stokes equation (BSNSEs for short).

$$\begin{cases} d\mathbf{v}(t) = \mu \mathbf{A}\mathbf{v}(t)dt + \mathbf{B}(\mathbf{v}(t))dt + \mathbf{g}(t)dt + Z(t)dW(t) \\ \mathbf{v}(T) = \xi, \ t \in [0,T], \ \mathbf{x} \in G. \end{cases} \tag{3.1}$$

The difference between the above system and the system in [18] is that the signs in front of $\mathbf{A}$ and $\mathbf{B}$ have been reversed. One sees this kind of change of signs in the theory of backward PDE in deterministic case. Here we reverse the signs, but we still consider the filtration forward in time so that the filtration still represents the information collected during the process of fluid flow. We believe this is better than the equation introduced in [18]. The existence and uniqueness of the system come more naturally than that of in [18].

Let $\{\mathbf{e}_i\}_{i=1}^{\infty}$ be the complete orthonormal system of H defined in the previous section. Without loss of generality, let us suppose that $\{\mathbf{e}_i\}_{i=1}^{\infty} \subset \mathrm{V}$. For any N in $\in \mathbb{N}$, let H_N =span $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_N\}$. Let P_N be the orthogonal projection from H to H_N, i.e., $\mathrm{H}_N = P_N\mathrm{H}$. Similarly, one defines V_N and $\mathrm{V'}_N$. It is clear that $\mathrm{V}_N = \mathrm{H}_N = \mathrm{V'}_N$ for all N. Under the projection P_N, let us construct a finite dimensional system and denote

$$\mathbf{A}^N = P_N\mathbf{A},\ \mathbf{B}^N = P_N\mathbf{B},\ \mathbf{g}^N = P_N\mathbf{g},$$

$$W^N(t) \triangleq P_N W(t) = \sum_{i=1}^{N} \sqrt{q_i} b^i(t)\mathbf{e}_i \text{ and } \xi^N = E(P_N\xi | \mathcal{F}_T^N),$$

where $\{\mathcal{F}_t^N\}$ pertains to the natural filtration of the process $W^N(t)$. For simplicity, we write $\mathcal{F}_t^N$ as $\mathcal{F}_t$ when the Galerkin approximation scheme is applied. Then the projected backward stochastic Navier-Stokes equation can be written as:

$$\begin{cases} d\mathbf{v}^N(t) = \mu\mathbf{A}^N\mathbf{v}^N(t)dt + \mathbf{B}^N(\mathbf{v}^N(t))dt + \mathbf{g}^N(t)dt + Z^N(t)dW^N(t) \\ \mathbf{v}^N(T) = \xi^N,\ t \in [0,T],\ \mathbf{x} \in G, \end{cases} \tag{3.2}$$

where $Z^N\colon [0,T] \times \Omega \to L(\mathrm{H}_N, \mathrm{H}_N)$.

Proposition 3.1. Assume that $\xi \in L^2_{\mathcal{F}_T}(\Omega,\mathrm{H})$ and $\mathbf{g} \in L^2(0,T,\mathrm{V}')$. If $(\mathbf{v}^N(t), Z^N(t))$ is an adapted solution for the projected system (3.2), then

$$\mathbf{v}^N \in L^2_{\mathcal{F}}(\Omega; L^\infty([0,T];\mathrm{H})) \cap L^2_{\mathrm{F}}(\Omega; L^2(0,T;\mathrm{V})), \text{ and}$$

$$Z^N \in L^2_{\mathcal{F}}(\Omega; L^2(0,T;L_Q)).$$

Moreover, there is a uniform bound of $\mathbf{v}^N$ for all $N \in \mathbb{N}$ in the above space.

Proof. By the coercivity hypothesis and Itô's formula, one gets

$$\begin{aligned} &|\mathbf{v}^N(t)|^2 + \int_t^T \|Z^N(s)\|_{L_Q}^2 ds + 2\int_t^T \langle \mu\mathbf{A}^N\mathbf{v}^N(s), \mathbf{v}^N(s)\rangle ds \\ &= |\xi^N|^2 - 2\int_t^T \langle \mathbf{g}^N(s), \mathbf{v}^N(s)\rangle ds - 2\int_t^T ((Z^N(s))^*(\mathbf{v}^N(s)), dW^N(s)), \end{aligned} \tag{3.3}$$

where $(Z^N)^*$ is the adjoint of Z^N. Taking the expectation and an application of the Gronwall inequality yield

$$\begin{aligned} &E|\mathbf{v}^N(t)|^2 + E\int_t^T \|Z^N(s)\|_{L_Q}^2 ds + 2\mu E\int_t^T \|\mathbf{v}^N(s)\|^2 ds \\ &\le E|\xi^N|^2 + \frac{1}{\mu}\int_t^T \|\mathbf{g}(s)\|_{\mathrm{V}'}^2 ds + \mu E\int_t^T \|\mathbf{v}^N(s)\|^2 ds. \end{aligned}$$

Hence

$$\sup_{0\le t\le T} E\,|\,\mathbf{v}^N(t)|^2\ + E\int_0^T \|Z^N(s)\|_{L_Q}^2 ds + E\int_0^T \|\mathbf{v}^N(s)\|^2 ds\ \le\ K$$

for some constant K. We first take the sup over [0,T] in (3.3) and then take the expectation. By applying the above inequality and the Burkholder-Davis-Gundy inequality, one gets

$$E \sup_{0\le t\le T} |\mathbf{v}^N(t)|^2$$

$$\le E|\xi^N|^2 + \frac{1}{\mu}\int_0^T \|\mathbf{g}(s)\|_{\mathrm{V}'}^2 ds$$

$$+ 2E \sup_{0\le t\le T} |\int_t^T ((Z^N(s))^*(\mathbf{v}^N(s)), dW^N(s))|$$

$$\le K + 4E \sup_{0\le t\le T} |\int_0^T ((Z^N(s))^*(\mathbf{v}^N(s)), dW^N(s))|$$

$$\le K + 4KE\{\int_0^T \|Z^N(s)\|_{L_Q}^2 |\mathbf{v}^N(s)|^2 dt\}^{\frac{1}{2}}$$

$$\le K + 4KE\{\sup_{0\le t\le T} |\mathbf{v}^N(s)|^2 \int_0^T \|Z^N(s)\|_{L_Q}^2 dt\}^{\frac{1}{2}}$$

$$\le K + \frac{1}{2} E \sup_{0\le t\le T} |\mathbf{v}^N(s)|^2 + KE\int_0^T \|Z^N(s)\|_{L_Q}^2 dt.$$

This completes the proof. Note that for notational simplicity, the letter K denotes different constants, which vary from line to line.

Based on the hypotheses, one can prove the following results easily.

Lemma 3.2. For all $\mathbf{u}, \mathbf{v} \in \mathrm{V}$ and *a* constant C_G depending on the domain G,

$$|\langle \mathbf{B}(\mathbf{u}) - \mathbf{B}(\mathbf{v}), \mathbf{u}-\mathbf{v}\rangle| \le \frac{\mu}{2}\|\mathbf{u}-\mathbf{v}\|^2 + \frac{C_G^2}{2\mu}|\mathbf{u}-\mathbf{v}|^2\|\mathbf{v}\|^2.$$

Corollary 3.3. For all $\mathbf{u}, \mathbf{v} \in L_{\mathrm{F}}^2(\Omega; L^2(0,T;\mathrm{V}))$, let $r_1(t) = \frac{C_G^2}{\mu}\int_0^t \|\mathbf{u}(s)\|^2 ds$ and $r_2(t) = \frac{C_G^2}{\mu}\int_0^t \|\mathbf{v}(s)\|^2 ds$. Then

$$\langle \mu\mathbf{A}\mathbf{w} + \mathbf{B}(\mathbf{u}) - \mathbf{B}(\mathbf{v}) + \frac{1}{2}\dot{r}_i(t)\mathbf{w}, \mathbf{w}\rangle \ge \frac{\mu}{2}\|\mathbf{w}\|^2 \ge 0, \qquad i = 1,2,$$

where $\mathbf{w} = \mathbf{u} - \mathbf{v}$.

Theorem 3.4. Assume that $\xi \in L_{\mathcal{F}_T}^2(\Omega,\mathrm{H})$ and $\mathbf{g} \in L^2(0,T;\mathrm{V}')$. Then the backward stochastic Navier-Stokes equation (3.1) admits an adapted solution $(\mathbf{v}(t), Z(t))$ in

$$\{L_{\mathcal{F}}^2(\Omega; L^\infty([0,T];\mathrm{H})) \cap L_{\mathcal{F}}^2(\Omega; L^2(0,T;\mathrm{V}))\} \times L_{\mathcal{F}}^2(\Omega; L^2(0,T;L_Q)).$$

Proof. The existence of an adapted solution $(\mathbf{v}(t), Z(t))$ can be proved along the lines of Theorem 4.7 in [18] with some variation. The integrability of the solution can be proved similar to Proposition 3.1.

Now let us prove the well-posedness of the backward stochastic Navier-Stokes system.

Theorem 3.5. Assume the conditions in Theorem 3.4. Then the solution of system (3.1) is unique and continuous with respect to the terminal condition and external body force.

Proof. Let $\xi_1, \xi_2 \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{H})$ and $\mathbf{g}_1, \mathbf{g}_2 \in L^2(0,T;\mathbb{V}')$. Let $(\mathbf{v}(t), Z(t))$ be the solution of (3.1) with respect to ξ_1 and $\mathbf{g}_1$, and let $(\mathbf{u}(t), Y(t))$ be the solution of (3.1) with respect to ξ_2 and $\mathbf{g}_2$. Define $\mathbf{w}(t) = \mathbf{v}(t) - \mathbf{u}(t)$ and $\sigma(t) = Z(t) - Y(t)$. Then

$$\begin{cases} d\mathbf{w}(t) = \mu\mathbf{A}\mathbf{w}(t)dt + (\mathbf{B}(\mathbf{v}(t)) - \mathbf{B}(\mathbf{u}(t)))dt + (\mathbf{g}_1(t) - \mathbf{g}_2(t))dt \\ \qquad + \sigma(t)dW(t) \\ \mathbf{w}(T) = \xi_1 - \xi_2. \end{cases}$$

Let us define $r(t) = \frac{C_G^2}{\mu}\int_0^t \|\mathbf{v}(s)\|^2 ds$. An application of the Itô formula to $e^{-r(t)}|\mathbf{w}(t)|^2$ and Corollary 3.3 implies

$$\begin{aligned} & e^{-r(t)}|\mathbf{w}(t)|^2 + \int_t^T e^{-r(s)}\|\sigma(s)\|^2_{L_Q} ds = e^{-r(T)}|\xi_1 - \xi_2|^2 \\ & -2\int_t^T e^{-r(s)}\{\langle \mu\mathbf{A}\mathbf{w}(s) + (\mathbf{B}(\mathbf{v}(s)) - \mathbf{B}(\mathbf{u}(s))) + \frac{1}{2}\dot{r}(s)\mathbf{w}, \mathbf{w}(s)\rangle\} ds \\ & -2\int_t^T e^{-r(s)}\langle \mathbf{g}_1(s) - \mathbf{g}_2(s), \mathbf{w}(s)\rangle\} ds - 2\int_t^T e^{-r(s)}((\sigma(s))^*(\mathbf{w}(s)), dW(s)) \\ & \le e^{-r(T)}|\xi_1 - \xi_2|^2 - \mu\int_t^T e^{-r(s)}\|\mathbf{w}(s)\|^2 ds \\ & + \frac{1}{\mu}\int_t^T \|\mathbf{g}_1(s) - \mathbf{g}_2(s)\|^2_{\mathbb{V}'} ds + \mu\int_t^T e^{-2r(s)}\|\mathbf{w}(s)\|^2 ds \\ & -2\int_t^T e^{-r(s)}((\sigma(s))^*(\mathbf{w}(s)), dW(s)). \end{aligned} \tag{3.4}$$

Taking the expectation, the above equality becomes

$$\begin{aligned} & \sup_{0\le t\le T} Ee^{-r(t)}|\mathbf{w}(t)|^2 + E\int_0^T e^{-r(s)}\|\sigma(s)\|^2_{L_Q} ds. \\ & +\mu E\int_0^T (e^{-r(s)} - e^{-2r(s)})\|w(s)\|^2 ds \\ & \le Ee^{-r(T)}|\xi_1 - \xi_2|^2 + \frac{1}{\mu}E\int_0^T \|\mathbf{g}_1(s) - \mathbf{g}_2(s)\|^2_{\mathbb{V}'} ds. \end{aligned}$$

On the other hand, taking sup in (3.4) and applying the Burkholder-Davis-Gundy inequality, one gets

$$\begin{aligned} & E\sup_{0\le t\le T} e^{-r(t)}|\mathbf{w}(t)|^2 + E\int_0^T e^{-r(s)}\|\sigma(s)\|^2_{L_Q} ds \\ & +\mu E\int_0^T (e^{-r(s)} - e^{-2r(s)})\|\mathbf{w}(s)\|^2 ds \\ & \le Ee^{-r(T)}|\xi_1 - \xi_2|^2 + \frac{1}{\mu}E\int_0^T \|\mathbf{g}_1(s) - \mathbf{g}_2(s)\|^2_{\mathbb{V}'} ds \\ & +2E\sup_{0\le t\le T}\left|\int_t^T e^{-r(s)}((\sigma(s))^*(\mathbf{w}(s)), dW(s))\right| \\ & \le Ee^{-r(T)}|\xi_1 - \xi_2|^2 + \frac{1}{\mu}E\int_0^T \|\mathbf{g}_1(s) - \mathbf{g}_2(s)\|^2_{\mathbb{V}'} ds \end{aligned}$$

$$+\frac{1}{2}E\sup_{0\le t\le T}|\mathbf{w}(t)|^2+KE\int_0^T e^{-2r(t)}\|\sigma(s)\|_{L_Q}^2dt.$$

Thus we have shown the continuity of solutions with respect to the terminal conditions and external body force. If $\xi_1=\xi_2$, P-a.s. and $\mathbf{g}_1=\mathbf{g}_2$, one deduces the uniqueness easily.

4. Projections and Estimates

Throughout this paper, we make the following assumptions:

(A1). $\mathbf{u}_0(\cdot)$is a deterministic function in $\mathbb{H}$;

(A2). $\mathbf{f}$: $[0,T]\to\mathbb{V}'$ is a deterministic smooth function such that for some positive constant C,

$$\|\mathbf{f}\|_{V'}\le C$$

for all $t\in[0,T]$, and $\mathbf{g}$: $\mathbb{V}\to\mathbb{V}'$ is a continuous operator with uniformly bounded first derivative;

(A3). g: $\mathbb{H}\to\mathbb{H}$ is a smooth convex operator such that its first derivative is bounded by a constant $L>0$. Also, there exists a positive constant $\alpha\in(0,1)$, such that $\mathrm{g}\in C^{2+\alpha}(\mathbb{H})$;

(A4). The operator σ: $\mathbb{H}\to L_2(\mathbb{H};\mathbb{H})$ is smooth. Here $L_2(\mathbb{H};\mathbb{H})$ denotes the class of Hilbert-Schmidt operators from $\mathbb{H}$ to $\mathbb{H}$. Also σ and its first derivative are bounded by a constant $L>0$.

Before solving the forward-backward system (2.3), we first go to the finite dimension and establish some a priori estimates. In order to apply the four step scheme introduced by Ma, Protter, and Yong in [10], we will further project the system to $\mathbb{R}^N$ for any $N\in\mathbb{N}$.

Similar to Section 3, we define the following finite-dimensional projections

$$\mathrm{g}^N=P_N\mathrm{g},\ \mathbf{f}^N=P_N\mathbf{f},\ \mathbf{u}_0^N=E(P_N\mathbf{u}_0|\mathcal{F}_0^N)\text{and }\sigma^N=P_N\sigma.$$

Then the projected forward-backward stochastic Navier-Stokes equation can be written as:

$$\begin{cases}d\mathbf{u}^N(t)=-\nu\mathbf{A}^N\mathbf{u}^N(t)dt-\mathbf{B}^N(\mathbf{u}^N(t))dt+\mathbf{f}^N(t)dt+\sigma^N(t,\mathbf{v}^N(t))dW^N(t)\\ d\mathbf{v}^N(t)=-\mu\mathbf{A}^N\mathbf{v}^N(t)dt-\mathbf{B}^N(\mathbf{v}^N(t))dt+\mathbf{g}^N(\mathbf{v}^N(t))dt+Z^N(t)dW^N(t)\\ \mathbf{u}^N(0,x)=\mathbf{u}_0^N(\mathbf{x}),\ \text{and }\mathbf{v}^N(T)=\mathrm{g}^N(\mathbf{u}^N(T)),\ t\in[0,T],\ \mathbf{x}\in G.\end{cases}\tag{4.1}$$

Proposition 4.1. Assume (A1)-(A4), and suppose that $(\mathbf{u}^N,\mathbf{v}^N,Z^N)$ is a solution of (4.1). Then

$$\mathbf{u}^N\in L^2_{\mathcal{F}}(L^\infty([0,T];\mathbb{H}))\cap L^2_{\mathcal{F}}(\Omega;L^2(0,T;\mathbb{V})),$$

$$\mathbf{v}^N\in L^\infty([0,T]\times\Omega;\mathbb{H})\cap L^2_{\mathcal{F}}(\Omega;L^2(0,T;\mathbb{V})),\ \text{and}$$

$$Z^N\in L^2_{\mathcal{F}}(\Omega;L^2(0,T;L_Q)).$$

Moreover, there is a uniform bound of $\mathbf{u}^N$ for all $N\in\mathbb{N}$ in the above space.

Proof. Since $\mathbf{u}_0\in\mathbb{H}$, $\mathbf{f}$ is uniformly bounded over t and ω, and σ is uniformly bounded on $[0,T]\times\mathbb{H}\times\Omega$, a direct application of results from [13] yields the integrability of $\mathbf{u}^N$. Similarly, based on the assumption on $\mathbf{g}$, g and the integrability of $\mathbf{u}^N$, an application of the Itô formula and the Gronwall lemma shows that $\mathbf{v}^N$ and Z^N lie in the desired spaces. The readers may refer to [18] for detailed proof.

In order to show the existence and uniqueness of the solutions to (4.1) for any $N \in \mathbb{N}$, we project the system to $\mathbb{R}^N$. For $1 \le i \le N$, let $\langle \mathbf{u}^N(t), \mathbf{e}_i \rangle = \hat{u}_i^N(t)$ and we denote $\hat{\mathbf{u}}^N(t) = \begin{pmatrix} \hat{u}_1^N(t) \\ \vdots \\ \hat{u}_N^N(t) \end{pmatrix} \in \mathbb{R}^N$. For $\mathbf{A}^N$, notice that $\langle \mathbf{A}^N \mathbf{u}^N(t), \mathbf{e}_i \rangle = \langle \sum_{k=1}^N \lambda_k \hat{u}_k^N(t) \mathbf{e}_k, \mathbf{e}_i \rangle = \lambda_i \hat{u}_i^N(t)$. By Corollary 2.3, we have

$$\|\mathbf{u}^N(t)\| = \langle \mathbf{A}\mathbf{u}^N(t), \mathbf{u}^N(t) \rangle^{\frac{1}{2}}$$

$$= \langle \sum_{j=1}^N \lambda_j \hat{u}_j^N(t) \mathbf{e}_j, \sum_{i=1}^N \hat{u}_i^N(t) \mathbf{e}_i \rangle^{\frac{1}{2}} = \sqrt{\sum_{j=1}^N \lambda_j (\hat{u}_j^N(t))^2}.$$

Hence, we define $\hat{\mathbf{A}}^N \hat{\mathbf{u}}^N(t) \triangleq \begin{pmatrix} \lambda_1 \hat{u}_1^N(t) \\ \lambda_2 \hat{u}_2^N(t) \\ \vdots \\ \lambda_N \hat{u}_N^N(t) \end{pmatrix}$. Since for $\mathbf{B}^N$, we have

$$\langle \mathbf{B}^N(\mathbf{u}^N(t)), \mathbf{e}_i \rangle = \langle P_N \mathbf{B}(\mathbf{u}^N(t)), \mathbf{e}_i \rangle$$

$$= \langle \sum_{j=1}^N \langle \mathbf{B}(\mathbf{u}^N(t)), \mathbf{e}_j \rangle \mathbf{e}_j, \mathbf{e}_i \rangle$$

$$= \langle \sum_{j=1}^N b(\mathbf{u}^N(t), \mathbf{u}^N(t), \mathbf{e}_j) \mathbf{e}_j, \mathbf{e}_i \rangle$$

$$= b(\mathbf{u}^N(t), \mathbf{u}^N(t), \mathbf{e}_i)$$

$$= b(\sum_{k=1}^N \hat{u}_k^N(t) \mathbf{e}_k, \sum_{l=1}^N \hat{u}_l^N(t) \mathbf{e}_l, \mathbf{e}_i)$$

$$= \sum_{k,l=1}^N b(\mathbf{e}_k, \mathbf{e}_l, \mathbf{e}_i) \hat{u}_k^N(t) \hat{u}_l^N(t),$$

we define

$$\hat{\mathbf{B}}^N(\hat{\mathbf{u}}^N(t)) \triangleq \begin{pmatrix} \sum_{k,l=1}^N b(\mathbf{e}_k, \mathbf{e}_l, \mathbf{e}_1) \hat{u}_k^N(t) \hat{u}_l^N(t) \\ \sum_{k,l=1}^N b(\mathbf{e}_k, \mathbf{e}_l, \mathbf{e}_2) \hat{u}_k^N(t) \hat{u}_l^N(t) \\ \vdots \\ \sum_{k,l=1}^N b(\mathbf{e}_k, \mathbf{e}_l, \mathbf{e}_N) \hat{u}_k^N(t) \hat{u}_l^N(t) \end{pmatrix}.$$

We also denote $\langle \mathbf{f}^N(t), \mathbf{e}_i \rangle = \hat{f}_i^N(t)$ and $\hat{\mathbf{f}}^N(t) \triangleq \begin{pmatrix} \hat{f}_1^N(t) \\ \vdots \\ \hat{f}_N^N(t) \end{pmatrix}$. Similarly, we can define $\hat{\mathbf{v}}^N(t)$ and $\hat{\mathbf{B}}^N(\hat{\mathbf{v}}^N(t))$.

For the initial condition, we define $\hat{\mathbf{u}}_0^N \triangleq \begin{pmatrix} \langle \mathbf{u}_0^N, \mathbf{e}_1 \rangle \\ \vdots \\ \langle \mathbf{u}_0^N, \mathbf{e}_N \rangle \end{pmatrix}$, and for the terminal condition, define

$$\hat{\mathbf{v}}^N(T)=\hat{\mathrm{g}}^N\big(\hat{\mathbf{u}}^N(T)\big)\triangleq\begin{pmatrix}\langle \mathrm{g}^N(\sum_{j=1}^{N}\hat{\mathbf{u}}_j^N(T)\mathbf{e}_j),\mathbf{e}_1\rangle\\ \langle \mathrm{g}^N(\sum_{j=1}^{N}\hat{\mathbf{u}}_j^N(T)\mathbf{e}_j),\mathbf{e}_2\rangle\\ \cdots\\ \langle \mathrm{g}^N(\sum_{j=1}^{N}\hat{\mathbf{u}}_j^N(T)\mathbf{e}_j),\mathbf{e}_N\rangle\end{pmatrix}.$$

Similarly, we define

$$\hat{\mathbf{g}}^N(\hat{\mathbf{v}}^N(t))\triangleq\begin{pmatrix}\langle \mathbf{g}^N(\sum_{j=1}^{N}\hat{\mathbf{v}}_j^N(t)\mathbf{e}_j),\mathbf{e}_1\rangle\\ \langle \mathbf{g}^N(\sum_{j=1}^{N}\hat{\mathbf{v}}_j^N(t)\mathbf{e}_j),\mathbf{e}_2\rangle\\ \cdots\\ \langle \mathbf{g}^N(\sum_{j=1}^{N}\hat{\mathbf{v}}_j^N(t)\mathbf{e}_j),\mathbf{e}_N\rangle\end{pmatrix}.$$

Since

$$\langle\int_t^T Z^N(s)dW^N(s),\mathbf{e}_i\rangle$$

$$=\langle\sum_{k=1}^{N}\sqrt{q_k}\int_t^T Z^N(s)(\mathbf{e}_k)db^k(s),\mathbf{e}_i\rangle$$

$$=\langle\sum_{k=1}^{N}\sqrt{q_k}\sum_{l=1}^{N}\int_t^T\langle Z^N(s)(\mathbf{e}_k),\mathbf{e}_l\rangle\mathbf{e}_l db^k(s),\mathbf{e}_i\rangle$$

$$=\sum_{k=1}^{N}\sqrt{q_k}\int_t^T\langle Z^N(s)(\mathbf{e}_k),\mathbf{e}_i\rangle db^k(s)$$

$$=\sum_{k=1}^{N}\int_t^T\langle\sqrt{q_k}\mathbf{e}_k,(Z^N(s))^*(\mathbf{e}_i)\rangle db^k(s)$$

$$=\sum_{k=1}^{N}\int_t^T\langle Q^{\frac{1}{2}}(\mathbf{e}_k),(Z^N(s))^*(\mathbf{e}_i)\rangle db^k(s),$$

we define $\hat{Z}^N(t)$ as

$$\begin{pmatrix}\langle Q^{\frac12}(\mathbf{e}_1),(Z^N(s))^*(\mathbf{e}_1)\rangle, & \langle Q^{\frac12}(\mathbf{e}_2),(Z^N(s))^*(\mathbf{e}_1)\rangle, & \cdots, & \langle Q^{\frac12}(\mathbf{e}_N),(Z^N(s))^*(\mathbf{e}_1)\rangle\\ \langle Q^{\frac12}(\mathbf{e}_1),(Z^N(s))^*(\mathbf{e}_2)\rangle, & \langle Q^{\frac12}(\mathbf{e}_2),(Z^N(s))^*(\mathbf{e}_2)\rangle, & \cdots, & \langle Q^{\frac12}(\mathbf{e}_N),(Z^N(s))^*(\mathbf{e}_2)\rangle\\ \vdots & \vdots & \vdots & \vdots\\ \langle Q^{\frac12}(\mathbf{e}_1),(Z^N(s))^*(\mathbf{e}_N)\rangle, & \langle Q^{\frac12}(\mathbf{e}_2),(Z^N(s))^*(\mathbf{e}_N)\rangle, & \cdots, & \langle Q^{\frac12}(\mathbf{e}_N),(Z^N(s))^*(\mathbf{e}_N)\rangle\end{pmatrix}$$

and $\widehat{\mathrm{W}}^N(t)\triangleq\begin{pmatrix}b^1(t)\\ \vdots\\ b^N(t)\end{pmatrix}$ where $\{b^j(t):1\le j\le N\}$ are N independent standard 1-dimensional Brownian motions. Similarly, we can define $\hat{\sigma}^N:\mathbb{R}^N\longrightarrow\mathbb{R}^{N\times N}$ as

$$\hat{\sigma}^N(t,\hat{\mathbf{v}}^N) \triangleq \begin{pmatrix} \langle \sigma^N(t,\sum_{k=1}^N \hat{\mathbf{v}}_k^N \mathbf{e}_k)(\mathbf{e}_1),\mathbf{e}_1\rangle, & \langle \sigma^N(t,\sum_{k=1}^N \hat{\mathbf{v}}_k^N \mathbf{e}_k)(\mathbf{e}_2),\mathbf{e}_1\rangle, & \cdots, & \langle \sigma^N(t,\sum_{k=1}^N \hat{\mathbf{v}}_k^N \mathbf{e}_k)(\mathbf{e}_N),\mathbf{e}_1\rangle \\ \langle \sigma^N(t,\sum_{k=1}^N \hat{\mathbf{v}}_k^N \mathbf{e}_k)(\boldsymbol{e}_1),\mathbf{e}_2\rangle, & \langle \sigma^N(t,\sum_{k=1}^N \hat{\mathbf{v}}_k^N \mathbf{e}_k)(\mathbf{e}_2),\mathbf{e}_2\rangle, & \cdots, & \langle \sigma^N(t,\sum_{k=1}^N \hat{\mathbf{v}}_k^N \mathbf{e}_k)(\mathbf{e}_N),\mathbf{e}_2\rangle \\ \vdots & \vdots & \vdots & \vdots \\ \langle \sigma^N(t,\sum_{k=1}^N \hat{\mathbf{v}}_k^N \mathbf{e}_k)(\boldsymbol{e}_1),\mathbf{e}_N\rangle, & \langle \sigma^N(t,\sum_{k=1}^N \hat{\mathbf{v}}_k^N \mathbf{e}_k)(\mathbf{e}_2),\mathbf{e}_N\rangle, & \cdots, & \langle \sigma^N(t,\sum_{k=1}^N \hat{\mathbf{v}}_k^N \mathbf{e}_k)(\mathbf{e}_N),\mathbf{e}_N\rangle \end{pmatrix}.$$

After defining $\hat{\sigma}^N$, we are ready to propose another assumption on σ:

(A5). There exists a positive continuous function $\nu(\cdot)$, such that for all $N \in \mathbb{N}$ and $(t,y) \in [0,T] \times \mathbb{R}^N$,

$$\nu(|y|)I \leq \hat{\sigma}^N(t,y)\hat{\sigma}^N(t,y)^T,$$

where I is the identity matrix in $\mathbb{R}^{N\times N}$.

Now (4.1) is equivalent to

$$\begin{cases} d\hat{\mathbf{u}}^N(t) = -\nu\widehat{\mathbf{A}}^N(u^N(t))dt - \widehat{\mathbf{B}}^N(\hat{\mathbf{u}}^N(t))dt + \hat{\mathbf{f}}^N(t)dt + \hat{\sigma}^N(t,\hat{\mathbf{v}}^N)d\widehat{\mathbf{W}}^N(t) \\ d\hat{\mathbf{v}}^N(t) = -\mu\widehat{\mathbf{A}}^N(v^N(t))dt - \widehat{\mathbf{B}}^N(\hat{\mathbf{v}}^N(t))dt + \hat{\mathbf{g}}^N(\hat{\mathbf{v}}^N(t))dt + \hat{Z}^N(t)d\widehat{\mathbf{W}}^N(t) \\ \hat{\mathbf{u}}^N(0,\mathbf{x}) = \hat{\mathbf{u}}_0^N(\mathbf{x}), \text{ and } \hat{\mathbf{v}}^N(T) = \hat{\mathrm{g}}^N(\hat{\mathbf{u}}^N(T)),\ t\in[0,T],\mathbf{x}\in G, \end{cases} \tag{4.2}$$

a system in $\mathbb{R}^N$.

5. Four Step Scheme

In this section, we are going to show the existence and uniqueness of the solution of (4.1) by means of four-step scheme. To achieve this goal, certain truncation and approximation are employed

Theorem 5.1. Assume that (A1)-(A5) holds. The projected system (4.1) has a unique adapted solution $(\mathbf{u}^N,\mathbf{v}^N,Z^N)$ in the space of

$$\{L^2_{\mathcal{F}}(L^\infty([0,T];\mathbb{H})) \cap L^2_{\mathrm{F}}(\Omega;L^2(0,T;\mathbb{V}))\}$$

$$\times \{L^\infty([0,T]\times\Omega;\mathbb{H}) \cap L^2_{\mathrm{F}}(\Omega;L^2(0,T;\mathbb{V}))\}$$

$$\times L^2_{\mathcal{F}}(\Omega;L^2(0,T;L_Q)).$$

Proof. Clearly, systems (4.1) and (4.2) are equivalent. We only need to show the existence of a unique adapted solution of system (4.2). For every $M \in \mathbb{N}$, let $L_{M,N}\colon \mathbb{R}^N \to \mathbb{R}$ be a smooth function with bounded first derivative such that

$$L_{M,N}(\mathbf{x}) = \begin{cases} 1 & \text{if } |\mathbf{x}| < M, \\ 0 & \text{if } |\mathbf{x}| > M+1, \\ 0 \leq L_{M,N}(\mathbf{x}) \leq 1 & \text{otherwise.} \end{cases}$$

Under $L_{M,N}$ one defines a truncated system:

$$\begin{cases} d\hat{\mathbf{u}}^{M,N}(t) = & -\nu\widehat{\mathbf{A}}^N(\hat{\mathbf{u}}^{M,N}(t))dt - L_{M,N}(\hat{\mathbf{u}}^{M,N}(t))\widehat{\mathbf{B}}^N(\hat{\mathbf{u}}^{M,N}(t))dt + \hat{\mathbf{f}}^N(t)dt \\ & +\hat{\sigma}^N(t,\hat{\mathbf{v}}^{M,N})d\widehat{\mathbf{W}}^N(t) \\ d\hat{\mathbf{v}}^{M,N}(t) = & -\mu\widehat{\mathbf{A}}^N(\hat{\mathbf{v}}^{M,N}(t))dt - L_{M,N}(\hat{\mathbf{v}}^{M,N}(t))\widehat{\mathbf{B}}^N(\hat{\mathbf{v}}^{M,N}(t))dt + \hat{\mathbf{g}}^N(\hat{\mathbf{v}}^{M,N}(t))dt \\ & +\hat{Z}^{M,N}(t)d\widehat{\mathbf{W}}^N(t) \\ \hat{\mathbf{u}}^{M,N}(0,\mathbf{x}) = & \hat{\mathbf{u}}_0^N(\mathbf{x}), \text{ and } \hat{\mathbf{v}}^{M,N}(T) = \hat{\mathrm{g}}^N(\hat{\mathbf{u}}^{M,N}(T)),\ t\in[0,T],\ \mathbf{x}\in G. \end{cases} \tag{5.1}$$

For fixed M, N and $x \in G$, let us call the above system the M -truncated system, and solve it in the following four steps.

Step 1: Define $z^{M,N}(t,y,p) \triangleq p\hat{\sigma}^N(t,y)$, $\forall (t,y,p) \in [0,T] \times \mathbb{R}^N \times \mathbb{R}^{N\times N}$.

Step 2: For $1 \le k \le N$, we need to solve the following parabolic system for $\theta(t,x)$: $[0,T] \times \mathbb{R}^N \mapsto \mathbb{R}^N$:

$$\begin{cases} \theta_t^k + \dfrac{1}{2} tr\{\theta_{xx}^k(t,x)\hat{\sigma}^N(\hat{\sigma}^N)^T(t,\theta)\} - \langle \nu\hat{\mathbf{A}}^N x + L_{M,N}(x)\hat{\mathbf{B}}^N(x) - \hat{\mathbf{f}}^N(t), \theta_x^k \rangle \\ \qquad + [\mu\hat{\mathbf{A}}^N\theta + L_{M,N}(\theta)\hat{\mathbf{B}}^N(\theta) - \hat{\mathbf{g}}^N(\theta)]^k = 0,\ (t,x) \in [0,T) \times \mathbb{R}^N \\ \qquad \theta(T,x) = \hat{g}^N(x),\ \forall x \in \mathbb{R}^N, \end{cases} \tag{5.2}$$

where $[\cdot]^k$ means the k -th component of $[\cdot]$. Denote

$$-\nu\hat{\mathbf{A}}^N x - L_{M,N}(x)\hat{\mathbf{B}}^N(x) + \hat{\mathbf{f}}^N(t) \text{ and}$$

$$-\mu\hat{\mathbf{A}}^N y - L_{M,N}(y)\hat{\mathbf{B}}^N(y) + \hat{\mathbf{g}}^N(y).$$

by $\varphi(t,x)$ and $\psi(t,y)$, respectively.

First of all, φ, ψ, $\hat{\sigma}^N$ and $\hat{g}^N$ are smooth functions taking values in $\mathbb{R}^N$, $\mathbb{R}^N$, $\mathbb{R}^{N\times N}$ and $\mathbb{R}^N$, respectively. Clearly, the first derivatives of $\hat{\mathbf{A}}^N$ and $\hat{\mathbf{g}}^N$ are bounded. Since $L_{M,N}(x)$ is the constant 0 when $|x| > M+1$, both $L_{M,N}$ and its first derivative are 0 outside $\{x: |x| > M+1\}$. Clearly $\hat{\mathbf{B}}^N$ is bounded inside $\{x: |x| > M+1\}$. Hence the first derivative of $L_{M,N}\hat{\mathbf{B}}^N$ is bounded by some constant C. Thus we have shown that φ and ψ have uniformly bounded first derivatives with respect to x and y. It is also clear that $\hat{\sigma}^N$ and $\hat{g}^N$ have uniformly bounded first derivatives with respect to x and y.

By assumption (A4), it is easy to see that

$$\left\|\hat{\sigma}^N(t,y)\right\| \le C$$

for some constant $C > 0$ and all $(t,y) \in [0,T] \times \mathbb{R}^N$. Together with assumption (A5), we know that

$$\nu(|y|)I \le \hat{\sigma}^N(t,y)\hat{\sigma}^N(t,y)^T \le CI$$

for all $(t,y) \in [0,T] \times \mathbb{R}^N$ and some positive continuous function $\nu(\cdot)$, where I is the identity matrix in $\mathbb{R}^{N\times N}$.

By assumption (A2), the definition of $\hat{\mathbf{A}}^N$, $\hat{\mathbf{B}}^N$ and $L_{M,N}$, one gets

$$|\varphi(t,x)| + |\psi(t,0)| \le C,$$

for some constant C and all $(t,x) \in [0,T] \times \mathbb{R}^N$. Also, it is clear that $\hat{g}^N$ is bounded in $C^{2+\alpha}(\mathbb{R}^N)$ for some $\alpha \in (0,1)$. Thus an application of the results in [10] and [12] implies that the parabolic system (5.2) admits a unique classical solution $\theta(t,x)$ which is bounded, and $\theta_t(t,x)$, $\theta_x(t,x)$ and $\theta_{xx}(t,x)$ are all bounded as will.

Step 3: Utilizing the functions θ obtained in step 2 and based on our assumptions, the existence and uniqueness of a solution to the following forward equation are guaranteed:

$$\begin{cases} d\hat{\mathbf{u}}^{M,N}(t) = \varphi(t,\hat{\mathbf{u}}^{M,N}(t))dt + \hat{\sigma}^N(t,\theta(t,\hat{\mathbf{u}}^{M,N}(t)))d\hat{\mathrm{W}}^N(t) \\ \qquad \mathbf{u}^{M,N}(0,\mathbf{x}) = \hat{\mathbf{u}}_0^N(\mathbf{x}),\ t \in [0,T]. \end{cases}$$

Step 4: Set

$$\begin{cases} \qquad \hat{\mathbf{v}}^{\boldsymbol{M,N}}(t) = \theta(t,\hat{\mathbf{u}}^{M,N}(t)) \\ \hat{Z}^{M,N}(t) = z^{M,N}(t,\theta(t,\hat{\mathbf{u}}^{M,N}(t)),\theta_x(t,\hat{\mathbf{u}}^{M,N}(t))). \end{cases}$$

The results in [10] and [12] guarantee that $(\hat{\mathbf{u}}^{M,N}, \hat{\mathbf{v}}^{M,N}, \hat{Z}^{M,N})$ is the unique adapted solution for M -truncated system (5.1).

Similar to the proof of Proposition 4.1, one can show that there exists some positive constant $C(N)$, independent of M, such that $E\{sup_{0\le t\le T} \ |\hat{\mathbf{u}}^{M,N}(t)|\} \le C(N)$ and $sup_{0\le t\le T} \ |\hat{\mathbf{v}}^{M,N}(t)| \le C(N)$. For $K \in \mathbb{N}$, define $A^K_{M,N} = \{\omega: sup_{0\le t\le T} \ |\hat{\mathbf{u}}^{M,N}(t)| < K\}$. For any $M > C(N)$ and all $\omega \in A^M_{M,N}$, clearly

$$L_{M,N}(\hat{\mathbf{u}}^{M,N}(t)) = L_{M,N}(\hat{\mathbf{v}}^{M,N}(t)) = 1.$$

Thus, system (5.1) is the same as system (4.2), and the sample path of the solutions of (5.1) and (4.2) agree on $A^M_{M,N}$ P-almost surely. Also

$$L_{M+1,N}(\hat{\mathbf{u}}^{M,N}(t)) = L_{M+1,N}(\hat{\mathbf{v}}^{M,N}(t)) = 1 \text{on } A^M_{M,N},$$

and $(\hat{\mathbf{u}}^{M,N}, \hat{\mathbf{v}}^{M,N}, \hat{Z}^{M,N})$ solves the $M+1$-truncated system on $A^M_{M,N}$. Hence

$$A^M_{M,N} \subset A^{M+1}_{M+1,N} \tag{5.3}$$

for all $M > C(N)$. To complete the proof, we only need to show that $A^M_{M,N} \to \Omega$ P-a.s. as M approaches ∞.

Similar to the proof of Proposition 4.1, it is easy to see that $\{\hat{\mathbf{u}}^{M,N}\}_{M=1}^{\infty}$ is uniformly bounded over M in $L^2_{\mathcal{F}}(\Omega; C([0,T]; \mathbb{R}^N))$, there exists $\hat{\mathbf{u}}^N$ in the space $L^2_{\mathcal{F}}(\Omega; C([0,T]; \mathbb{R}^N))$, and a subsequence $\{\hat{\mathbf{u}}^{M_n,N}\}_{n=1}^{\infty}$, such that

$$\lim_{n\to\infty} E\ \{\sup_{0\le t\le T} |\hat{\mathbf{u}}^{M_n,N}(t) - \hat{\mathbf{u}}^N(t)|\} = 0\,.$$

So

$$\lim_{n\to\infty} P\ \{\omega: \sup_{0\le t\le T} |\hat{\mathbf{u}}^{M_n,N}(t) - \hat{\mathbf{u}}^N(t)| \le \varepsilon\} = 1 \tag{5.4}$$

for any $\varepsilon > 0$, and

$$\lim_{K\to\infty} P\ \{\omega: \sup_{0\le t\le T} |\hat{\mathbf{u}}^N(t)| \le K\} = 1. \tag{5.5}$$

Since

$$\sup_{0\le t\le T} |\hat{\mathbf{u}}^{M_n,N}(t)| \le \sup_{0\le t\le T} |\hat{\mathbf{u}}^{M_n,N}(t) - \hat{\mathbf{u}}^N(t)| + \sup_{0\le t\le T} |\hat{\mathbf{u}}^N(t)|,$$

we have

$$\{\omega:\ \sup_{0\le t\le T} |\hat{\mathbf{u}}^{M_n,N}(t) - \hat{\mathbf{u}}^N(t)| + \sup_{0\le t\le T} |\hat{\mathbf{u}}^N(t)| \le M_N\ \} \subset A^{M_n}_{M_n,N}$$

Hence

$$\{\omega:\ \sup_{0\le t\le T} |\hat{\mathbf{u}}^{M_n,N}(t) - \hat{\mathbf{u}}^N(t)| \le \frac{1}{2}\} \cap \{\omega:\ \sup_{0\le t\le T} |\ \hat{\mathbf{u}}^N(t)| \le\ M_n\ - \frac{1}{2}\}$$
$$\subset A^{M_n}_{M_n,N}.$$

Together with (5.3), (5.4) and (5.5), it is clear that $A^{M_n}_{M_n,N}$ monotonically increases to an almost sure set. Since the sample path of the solutions of (5.1) and (4.2) agree P-almost surely on $A^{M_n}_{M_n,N}$, we have shown that the limit of $(\hat{\mathbf{u}}^{M_n,N}, \hat{\mathbf{v}}^{M_n,N}, \hat{Z}^{M_n,N})$ as n approaches infinity is a solution of system (4.2). Let us denote it by $(\hat{\mathbf{u}}^N, \hat{\mathbf{v}}^N, \hat{Z}^N)$. Since systems (4.1) & (4.2) are equivalent, we have shown the existence of an adapted solution of the system (4.1). The uniqueness of the solution follows from the uniqueness of the solution of the system (5.1). The integrability of the solution is followed by Proposition 4.1.

6. Galerkin Approximation

In this section, we are going to solve the forward-backward system (2.3) via the Galerkin approximation scheme credited to the Russian mathematician Galerkin.

Theorem 6.1. Assume (A1)-(A5). The forward-backward stochastic Navier-Stokes equation (2.3) admits a unique adapted solution $(\mathbf{u}, \mathbf{v}, Z)$ in the space

$$\{L^2_{\mathcal{F}}(\Omega; C^0(0,T;\mathbb{H})) \cap L^2_{\mathcal{F}}(\Omega; L^2(0,T;\mathbb{V}))\}$$

$$\times \{L^\infty(\Omega \times [0,T];\mathbb{H}) \cap L^2_{\mathcal{F}}(\Omega; L^2(0,T;\mathbb{V}))\}$$

$$\times L^2_{\mathcal{F}}(\Omega; L^2(0,T;L_Q)).$$

Proof. By Theorem 5.1, for any $N \in \mathbb{N}$, a unique adapted solution to the projected system (4.1), $(\mathbf{u}^N, \mathbf{v}^N, Z^N)$, with suitable integrability is guaranteed under assumptions (A1)-(A5).

Let us first look at the backward component of the system (4.1):

$$\begin{cases} d\mathbf{v}^N(t) = -\mu \mathbf{A}^N \mathbf{v}^N(t)dt - \mathbf{B}^N(\mathbf{v}^N(t))dt + \mathbf{g}^N(\mathbf{v}^N(t))dt + Z^N(t)dW^N(t) \\ \mathbf{v}^N(T) = \mathbf{g}^N(u^N(T)),\ t \in [0,T]. \end{cases}$$

Before solving this system, we need to show that the limit of $\mathrm{g}^N(\mathbf{u}^N(T))$ is indeed $\mathrm{g}(\mathbf{u}(T))$. It is clear that

$$\begin{aligned} &\|\mathrm{g}^N(\mathbf{u}^N(T)) - \mathrm{g}(\mathbf{u}(T))\| \\ &= \|\mathrm{g}^N(\mathbf{u}^N(T)) - \mathrm{g}^N(\mathbf{u}(T))\| + \|\mathrm{g}^N(\mathbf{u}(T)) - \mathrm{g}(\mathbf{u}(T))\| \\ &\le \|\mathrm{g}(\mathbf{u}^N(T)) - \mathrm{g}(\mathbf{u}(T))\| + \|\mathrm{g}^N(\mathbf{u}(T)) - \mathrm{g}(\mathbf{u}(T))\| \end{aligned}$$

By the definition of the Leray-Hodge projection, $\|\mathrm{g}^N(\mathbf{u}(T)) - \mathrm{g}(\mathbf{u}(T))\|$ converges to 0 as N approaches infinity. Since g is continuous, we also know that $\|\mathrm{g}(\mathbf{u}^N(T)) - \mathrm{g}(\mathbf{u}(T))\|$ converges to 0 as N approaches infinity. Thus we have shown that maybe along a subsequence, $\mathbf{v}^N(T)$ converges to $\mathbf{v}(T)$ in $L^\infty(\Omega;\mathbb{V})$ as N approaches infinity.

By assumption (A3), $\|\mathrm{g}^N(\mathbf{u}^N(T))\|$ for all $N \in \mathbb{N}$ and $\|\mathrm{g}(\mathbf{u}(T))\|$ are all uniformly bounded. Hence applying Theorem 4.6 and Theorem 5.2 in [18] under assumption (A6) and (A7), we know that as N approaches infinity, the limit of $(\mathbf{v}^N, Z^N)$, denoted by $(\mathbf{v}, Z)$, solves

$$\begin{cases} d\mathbf{v}(t) = -\mu \mathbf{A}\mathbf{v}(t)dt - \mathbf{B}(\mathbf{v}(t))dt + \mathbf{g}(\mathbf{v}(t))dt + Z(t)dW(t) \\ \mathbf{v}(T) = \mathrm{g}(\mathbf{u}(T)),\ t \in [0,T]. \end{cases}$$

The solution $(\mathbf{v}, Z)$ is in

$$\{L^\infty(\Omega \times [0,T];\mathbb{H}) \cap L^4_{\mathcal{F}}(\Omega; L^2(0,T;\mathbb{V}))\} \times L^2_{\mathcal{F}}(\Omega; L^2(0,T;L_Q)).$$

Now let us study the forward component of the system (4.1):

$$\begin{cases} d\mathbf{u}^N(t) = -\mathbf{v} A^N \mathbf{u}^N(t)dt - \mathbf{B}^N(\mathbf{u}^N(t))dt + \mathbf{f}^N(t)dt + \sigma^N(t, \mathbf{v}^N(t))dW^N(t) \\ \mathbf{u}^N(0,x) = \mathbf{u}_0^N(\mathbf{x}),\ t \in [0,T],\ \mathbf{x} \in G. \end{cases}$$

Under our assumptions, similar to the proof of Proposition 3.2 and Proposition 3.3 in [13], one gets that as N approaches infinity, the limit of $\mathbf{u}^N$, denoted by $\mathbf{u}$, solves

$$\begin{cases} d\mathbf{u}(t) = -\nu \mathbf{A}\mathbf{u}(t)dt - \mathbf{B}(\mathbf{u}(t))dt + \mathbf{f}(t)dt + \sigma(t, \mathbf{v}(t))dW(t) \\ \mathbf{u}(0,\mathbf{x}) = \mathbf{u}_0(\mathbf{x}),\ t \in [0,T],\ \mathbf{x} \in G. \end{cases}$$

The solution $\mathbf{u}$ is in

$$L^2_{\mathcal{F}}(\Omega; C^0(0,T;\mathbb{H})) \cap L^2_{\mathcal{F}}(\Omega; L^2(0,T;\mathbb{V})).$$

Putting all the pieces together, we have finished the proof of this theorem.

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